MASTER IN ACTUARIAL SCIENCE

SURVIVAL MODELS AND LIFE CONTINGENCIES

LECTURE NOTES (II)

SURVIVAL MODELS AND LIFE CONTINGENCIES

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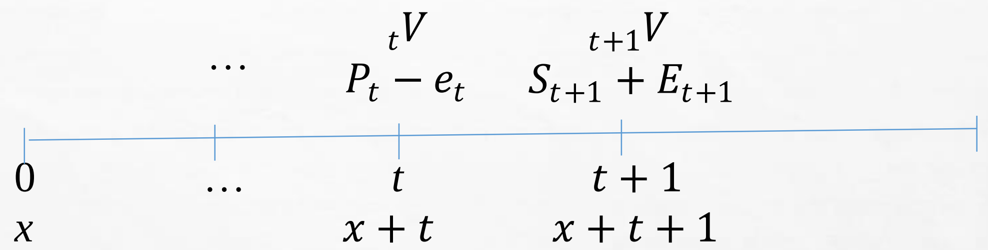
LECTURE 17



$$({}_tV + P_t - e_t)(1 + i_t) = q_{[x]+t}\,(S_{t+1} + E_{t+1}) + p_{[x]+t} \times {}_{t+1}V$$

$$\Leftrightarrow {}_{t+1}V = \frac{({}_tV + P_t - e_t)(1 + i_t) - q_{[x]+t}(S_{t+1} + E_{t+1})}{p_{[x]+t}}$$

Using this last equation, ${}_{t+1}V$ can be obtained from ${}_tV$.

More accurately the interest rate in year $[t, t+1[$ should be denoted $i_{[t,t+1[}$.

DEATH STRAIN AT RISK (DSAR)

EXPECTED DEATH STRAIN (EDS)

ACTUAL DEATH STRAIN (ADS)

Def. 31: The policyholder may survive year $t+1$ or die in it. If the policyholder has died in the year, the insurer must provide the extra amount to increase the policy value to the death benefit payable, plus any related expense; this extra amount required to increase the policy value to the death benefit is called the **Death Strain At Risk** (**DSAR**), or the **Sum at Risk** or the **Net Amount at Risk**, at time $t+1$. It is a random variable.

$$\text{DSAR} = \begin{cases} 0, & \text{if the life survives to } t+1 \quad (\text{with probability } p_{x+t}) \\ S_{t+1} + E_{t+1} - {}_{t+1}V, & \text{if the life dies in the year } [t, t+1[\ (\text{wp } q_{x+t}) \end{cases}$$

q_{x+t} *is* the probability of claiming in the policy year *t* to *t* + 1.

The expected amount of the death strain is called the **Expected Death Strain**:

$$\text{EDS} = E[DSAR] = 0 + q_{x+t}\left(S_{t+1} + E_{t+1} - {}_{t+1}V\right).$$

The **actual death strain (ADS)** is the observed value at t+1 of the death strain random variable (we already know what happened):

$$\text{ADS} = \begin{cases} 0, & \text{if the life survived to } t+1 \\ S_{t+1} + E_{t+1} - {}_{t+1}V, & \text{if the life died in the year } [t, t+1] \end{cases}$$

The **mortality profit** is defined as the difference **EDS-ADS.**

Remark 32: DSAR is an important measure of the insurer's risk if mortality exceeds the basis assumption, and is useful in determining risk management strategy, including reinsurance.

DSAR is a **random variable**: the maximum 'loss' the life office will make if the person dies in the next year.

EDS is an **expected value:** the amount that the life insurance company expects to pay extra to the year end reserve for the policy.

$$\mathbf{EDS} = \mathbf{E}[\mathbf{DSAR}].$$

ADS is the **observed value** at t+1 of the death strain random variable.

Mortality profit = EDS-ADS.

4.5.1.3 ***Annual profit***

Consider a group of identical policies issued at the same time. The recursive formulae for policy values show that *if* all cash flows between t and $t+1$ are as specified in the policy value basis, then the insurer will be in a break-even position (balanced) at time $t+1$, given that it was in a break-even position at time t. These cash flows depend on mortality, interest, expenses and, for participating policies, bonus rates.

Remark 33: In practice, it is very unlikely that all the assumptions will be met in any one year. If the assumptions are not met, one of two situations takes place:

1. The value of the insurer's assets at time $t+1$ is more than sufficient to pay benefits due at that time and to provide a policy value of ${}_{t+1}V$ for those policies still in force: **the insurer made a profit in the year.**

2. The insurer's assets at time $t+1$ are not sufficient to pay benefits due at that time and to provide a policy value of ${}_{t+1}V$ for those policies still in force: **the insurer made a loss in the year.**

Remark 34: In general terms,

Sources of profit:

- Actual interest earned on investments greater than the interest assumed in the policy value basis;
- Actual mortality less than the mortality assumed in the policy value basis for whole life and term and endowment policies; Actual mortality higher than the mortality assumed in the policy value basis for pure endowment and annuity policies;
- Actual expenses less than the expenses assumed in the policy value basis;
- Actual bonus or dividend less than the assumed in the policy value basis.

Sources of loss?

Rule: it is necessary to avoid ‘double counting’.

Sometimes the split is calculated in the order: interest, expenses, mortality.

At each step it is assumed that:

Factors not yet considered are as specified in the policy value basis;

Factors already considered are as actually occurred.

This avoids ‘double counting’ and gives the correct total.

Remark 35: the exercise of breaking down the profit or loss into its component parts is called **analysis of surplus**, and it is an important exercise after any valuation.

The analysis of surplus:

Will indicate if any parts of the valuation basis are too conservative or too weak;

Will assist in assessing the performance of the various managers involved in the business, and in determining the allocation of resources;

Will help to determine how much surplus should be distributed for participating business.

Study examples 7.3 and 7.8.

In pure endowment contracts, although the definition is the same, particular attention is required as there is only one possible time for the payment of the benefit - on the maturity of the contract.

Example:

A life insurance company issued 10-year pure endowment contracts to females aged 50 exact. The sum assured is 80 000, payable on maturity. Level quarterly premiums are payable in advance.

Basis:

Mortality: AM92 select

Interest: 6% per annum

a) calculate the death strain at risk at the end of the second year of the policies.

b) the company sold 4000 policies. During the first policy year there were five deaths. Calculate the minimum number of deaths during the second year for the company to experience a mortality profit during this year. Explain why it is a “minimum”.

a) Annual Premium

$$P\ddot{a}^{(4)}_{[50]:\overline{10}|} = 80\,000\,{}_{10}p_{[50]}v^{10} \Leftrightarrow P = \frac{80\,000\,{}_{10}p_{[50]}v^{10}}{\ddot{a}^{(4)}_{[50]:\overline{10}|}} = \frac{80\,000(0.5342964)}{\underbrace{\ddot{a}_{[50]:\overline{10}|}}_{7.698} - \frac{3}{8}(1 - 0.22578)} = 5681.47$$

$$\text{DSAR} = \begin{cases} 0, & \text{if the life survives to } t+1 \quad (\text{with probability } p_{x+t}) \\ S_{t+1} + E_{t+1} - {}_{t+1}V, & \text{if the life dies in the year } [t, t+1[\ (\text{wp } q_{x+t}) \end{cases}$$

Reserve at the end of the second year

$${}_2V = 80\,000\,{}_8p_{52}v^8 - 5681.47\ddot{a}^{(4)}_{52:\overline{8}|} = 48\,253.5123 - 5681.47(6.351) = 12\,170.50$$

Then

$$\text{DSAR} = \begin{cases} 0, & \text{if the life survives to } t = 2 \ (\text{wp } p_{[50]+1}) \\ \underbrace{0}_{S_2} + \underbrace{0}_{E_2} - \underbrace{12\,170.50}_{{}_2V}, & \text{if the life dies in the year } [1,2[\quad (\text{wp } q_{[50]+1}) \end{cases}$$

b)

Mortality profit/loss = Expected Death Strain – Actual Death Strain

$$\text{Expected Death Strain} = (4000-5)\underbrace{q_{[50]+1}}_{0.000569} \times (-12\,170.50) = -27\,665.425$$

$d =$ number of deaths during the second year

$$\text{Actual Death Strain} = (-12\,170.50)d$$

$$\text{mortality profit} = -27\,665.425 - (-12\,170.50d) > 0 \Leftrightarrow d > 2.27.$$

The minimum number of deaths during the second year for the company to experience a mortality profit during this year is three. “Minimum” because in pure endowment contracts, the least policyholders survive to maturity, the better to the insurer.

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Valuation <u>at premium dates</u> for policies with discrete cash flows at the <u>start/end of each year</u>

$$ {}_tV = E[L_t] = q_{[x]+t}(1+i_t)^{-1}(S_{t+1}+E_{t+1}) + p_{[x]+t}(1+i_t)^{-1}E[L_{t+1}] + e_t - P_t $$

Recursive formula

$$ {}_tV = q_{[x]+t}(1+i_t)^{-1}(S_{t+1}+E_{t+1}) + p_{[x]+t}(1+i_t)^{-1}\,{}_{t+1}V - (P_t - e_t) \Leftrightarrow $$

$$ {}_{t+1}V = \frac{{}_tV + (P_t - e_t) - (1+i_t)^{-1}\, q_{[x]+t}(S_{t+1}+E_{t+1})}{(1+i_t)^{-1}\, p_{[x]+t}} $$

Valuation <u>at premium dates</u> for policies with premium payments <u>at discrete intervals other than annually</u> is performed in the same way, since the definition still applies. Extra care is necessary to derive recursive formulae because the premiums and benefits may be paid with different frequency.

Valuation <u>between premium dates</u> for policies with premium payments at discrete intervals, annually and other than annually

The policy value is still the EPV of future benefits plus expenses minus premiums. A reasonable approximation is obtained by interpolating between the policy value **just after the previous premium** and the policy value **just before the next premium**.

<u>Annual case</u>: assume premiums are payable at t and $t+1$ and is necessary to calculate the policy value at $t+s, 0<s<1$. Interpolating between ${}_{t^+}V = {}_tV + P_t - e_t$ and ${}_{t+1}V$ solves the problem. The result of the interpolation is ${}_{t+s}V \approx {}_{t^+}V \times (1-s) + {}_{t+1}V \times s$.

<u>Cases other than annual</u>: assume premiums are payable at $t+k$ and $t+2k$, and is necessary to calculate the policy value at $t+k+s, 0<s<k<1$. Now, interpolating between ${}_{t+k^+}V = {}_{t+k}V + P_{t+k} - e_{t+k}$ and ${}_{t+2k}V$ solves the problem. The result is ${}_{t+k+s}V \approx {}_{t+k^+}V\left(1-\frac{s}{k}\right) + {}_{t+2k}V\left(\frac{s}{k}\right)$.

Valuation for policies with <u>continuous cash flows</u> using <u>first principles</u> and <u>Thiele's differential equation</u>

The definitions and procedures extend to policies where regular payments – premiums and/or annuities – are payable continuously and sums insured are payable immediately on death. It follows then that

$$ {}_tV = \int_t^\infty \frac{v(r)}{v(t)}(S_r + E_r)\ {}_{r-t}p_{[x]+t} \times \mu_{[x]+r}\, dr - \int_t^\infty \frac{v(r)}{v(t)}(P_r - e_r)\ {}_{r-t}p_{[x]+t}\, dr, $$

an equality that can be used to calculate ${}_tV$ by numerical integration.

As numerical integration is sometimes difficult, an alternative process is used, which requires this equality to be turned into a differential equation (Thiele's differential equation), a result that is obtained by means of a number of simple algebraic operations.

$$ \frac{d}{dt}\left({}_tV\right) = \delta_t\ {}_tV + P_t - e_t - \left(S_t + E_t - {}_tV\right)\mu_{[x]+t} $$

There are numerical techniques to solve differential equations (and obtain ${}_tV$).

4. CALCULATION OF PREMIUMS AND RESERVES (Dickson et al. – Chaps. 6-7, pp. 142-229)

4.7 Policy alterations

In practice, sometimes policyholders request a change in the terms of the policy, after it has been in force for some time:

- reducing or increasing premiums;
- changing the amount of the benefits;
- converting a whole life insurance to an endowment insurance;
- converting a non-participating policy to a with-profit policy;
- …

The common feature of these changes is that they are **requested by the policyholder** and were not part of the original terms of the policy.

Remark 43: Typical changes:

1. The policyholder wishes to cancel the policy with immediate effect. If the policy has a significant investment component, at least part of the funds belong to the policyholder and the insurer should pay a lump sum, the **surrender value (**or **cash value**)

Such a policy is said to **lapse (**when no surrender value is paid) or to be **surrendered (**when there is a return of assets of some amount to the policyholder).

The allowance for zero cash values for early surrenders reflects the need of the insurers to recover the new business strain associated with issuing the policy.

2. The policyholder wishes to pay no more premiums but does not want to cancel the policy.

A (reduced) sum insured is still payable on death or on survival to the end of the original term.

Any policy for which no further premiums are payable is said to be **paid-up**, and the reduced sum insured for a policy which becomes paid-up before the end of its original premium paying term is called a **paid-up sum insured**.

A whole life policy may be converted to a paid-up term insurance policy for the original sum insured.

Let C_t denote the cash surrender value at duration t.

Starting points for the calculation of an appropriate value for C_t could be:

(i) the policy value at t, ${}_tV$, if it is to be calculated in advance;

(ii) or the policy's asset share, AS_t, when the surrender value is not pre-specified.

AS_t is (approximately) equal to the cash the insurer actually has and ${}_tV$ is the amount the insurer should have at time t in respect of the original policy - if the policy value basis is close to the actual experience, then ${}_tV$ will be numerically close to AS_t.

C_t is usually less than 100% of either AS_t or ${}_tV$ and may include an explicit allowance for the expense of making the alteration, because:

1. The policyholder may be acting on knowledge that is not available to the insurer. For example, a policyholder may alter a whole life policy to a term insurance (with lower premiums or a higher sum insured) if he or she becomes aware that his or her health is failing. This is called **anti-selection** or **selection against the insurer**.
2. The insurance company will incur some expenses in making the alterations to the policy, and even in calculating and informing the policyholder of the revised values, which the policyholder may not agree to accept.
3. The alteration may cause the insurance company to realize assets it would otherwise have held, especially if the alteration is a surrender. This **liquidity risk** may lead to reduced investment returns for the company.

For alterations other than cash surrenders, C_t is a single premium, or an extra preliminary premium, for the future benefits. Together with the cash currently available C_t the future premiums are expected to provide the future benefits and pay for the future expenses. The equation of value for the altered benefits is then

$$C_t + \text{EPV at } t \text{ of future premiums, altered contract}$$

$$= \text{ EPV at } t \text{ of future benefits plus expenses, altered contract } (7.15)$$

The numerical value of the revised benefits and/or premiums calculated using equation (7.15) depends on the basis used for the calculation (survival model, interest rate, expenses, and future bonuses - for a with profits policy).

This basis may be the same as the premium basis, or the same as the policy value basis, but in practice usually differs from both of them.

Examples 7.13 and 7.14

4.8 Retrospective policy value

The policy value is sometimes called a *prospective policy value*. A *retrospective policy value* at duration *t*, could also be calculated, by accumulating premiums received less benefits paid up to time *t* for a large group of identical policies, *assuming the experience follows precisely the assumptions in the policy value basis*, and sharing the resulting fund equally among the surviving policyholders.

Under the usual conditions (the premium is calculated using the equivalence principle and the expected value of the future loss random variable is calculated using the premium basis) the retrospective and prospective policy values are equal.

CT5 September 2012 Q9

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5. MULTI-STATE POLICIES (Dickson *et al.* – Chap. 8, pp. 230-289; pictures below are from this book)

5.1 Multiple state models (MSM)

MSM are a recent development in actuarial science.

MSM simplify some traditional actuarial techniques.

Each state represents the status of an individual or a set of individuals.

The MSM introduced below are all extremely useful in an insurance context.

5.1.1 The alive–dead model

Consider (x). To model the uncertainty over the duration of the individual's future lifetime this is regarded as a random variable, T_x, with a given cumulative distribution function, $F_x(t) = P(T_x \leq t)$, and survival function, $S_x(t) = 1 - F_x(t)$.

MSM approach:
The individual aged $x \geq 0$ at time $t = 0$ is, at any time, in one of two **states**, 'Alive' – State 0 – and 'Dead' – State 1. Transition from state 0 to state 1 is allowed, but transitions in the opposite direction cannot occur. This is an example of a **multiple state model** with two states.

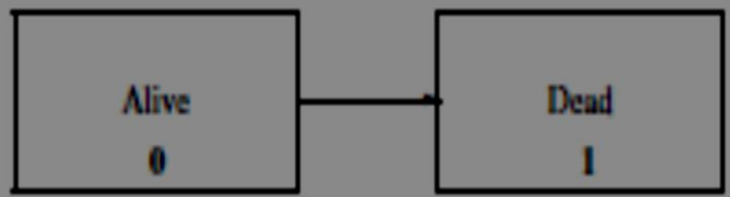


Figure 8.1 The alive–dead model.

For each $t \geq 0$ define a random variable $Y(t)$ so that

$$Y(t) = \begin{cases} 0 & \text{if the individual is alive at time } t \text{ (and aged } x+t) \\ 1 & \text{if the individual is dead at time } t \end{cases}$$

The set of random variables $\{Y(t)\}_{t\geq 0}$ is a **continuous time stochastic process (**a collection of random variables indexed by a continuous time variable). T_x is the time at which $Y(t)$ jumps from 0 to 1, that is, $T_x = max\{t: Y(t) = 0\}$.

The alive–dead model captures all the survival/mortality information for an individual that is necessary to find premiums and reserves for policies where payments – premiums, benefits and expenses – depend only on whether the individual is alive or dead at any given age, for example a term insurance or a whole life annuity.

Each one of the following models is appropriate for a given insurance policy in the sense that the condition for a payment relating to the policy, for example a premium, an annuity or a sum insured, is either that the individual is in a specified state at that time or that the individual makes an instantaneous transfer between a specified pair of states at that time.

5.1.2 Term insurance with increased benefit on accidental death (the accidental death model)

Suppose a term insurance policy under which the death benefit is $S1$ if death is due to an accident during the policy term and $S2 < S1$ if it is due to any other cause. The alive–dead model is not sufficient for this policy since, when the individual dies it is necessary to know whether death was due to an accident.

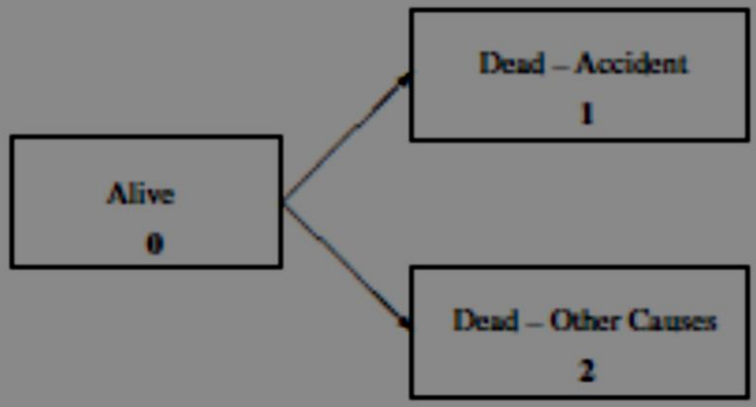


Figure 8.2 The accidental death model.

Now, for each $t \geq 0$ define a random variable $Y(t)$ so that

$$Y(t) = \begin{cases} 0 & \text{if the individual is alive at time } t \text{ (and aged } x+t\text{)} \\ 1 & \text{if the individual is dead (from an accident) at time } t \\ 2 & \text{if the individual is dead (from other causes) at time } t \end{cases}$$

- In these two models an individual starts in state 0, and, at some future time, dies.

- The difference is that we now need to distinguish between deaths due to accident and deaths due to other causes since the sum insured is different in the two cases.

- It is the benefits provided by the insurance policy which determine the nature of the appropriate model.

5.1.3 The permanent disability model

This is a model appropriate for a policy providing some or all of the following benefits:

- an annuity while permanently disabled,
- a lump sum on becoming permanently disabled,
- a lump sum on death.

Premiums are payable while healthy.

States: ‘Healthy’ – State 0; ‘Disabled’ – State 1 and ‘Dead’– State 2

An important feature is that disablement is permanent – there is no arrow from state 1 back to state 0.

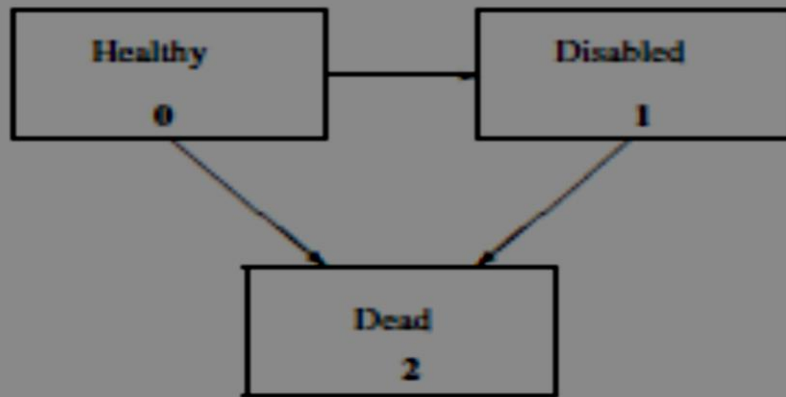


Figure 8.3 The permanent disability model.

$$Y(t) = \begin{cases} 0 & \text{if the individual is healthy at time } t \text{ (and aged } x+t\text{)} \\ 1 & \text{if the individual is permanently disabled at time } t \text{(and aged } x+t\text{)} \\ 2 & \text{if the individual is dead at time } t \end{cases}$$

5.1.4 The disability income insurance model

Disability income insurance pays a benefit during periods of sickness; the benefit ceases on recovery. This is an appropriate model for a policy which provides an annuity while the person is sick, with premiums payable while the person is healthy. It could also be used when there are lump sum payments on becoming sick or dying.

States: ‘Healthy’ – State 0; ‘Sick’ – State 1 and ‘Dead’– State 2.

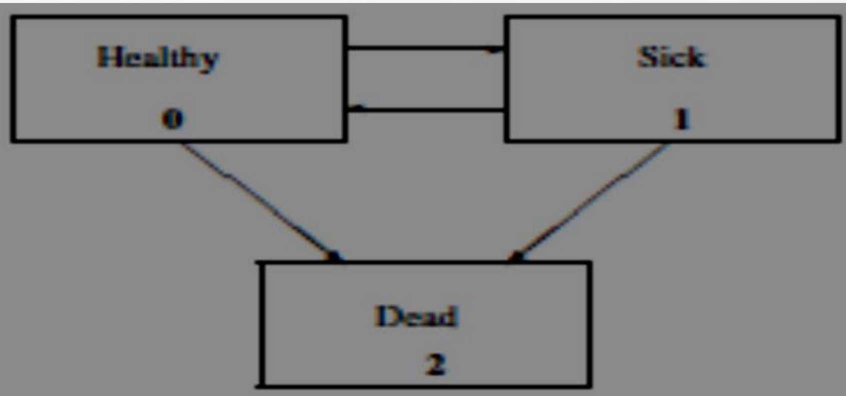


Figure 8.4 The disability income insurance model.

$$Y(t) = \begin{cases} 0 & \text{if the individual is healthy at time } t \text{ (and aged } x+t) \\ 1 & \text{if the individual is sick at time } t \text{(and aged } x+t) \\ 2 & \text{if the individual is dead at time } t \end{cases}$$

The model differs from the previous one because now it is possible to transfer from state 1 to state 0, that is, to recover from an illness.

This model also illustrates an important general feature of multiple state models which was not present before: the possibility of entering one or more states many times. This means that several periods of sickness could occur before death, with healthy (premium paying) periods in between.

5.1.5 The joint life and last survivor models

Def. 1: A **joint life annuity** is an annuity payable until the first death among a group of lives.

Def. 2: A **last survivor annuity** is an annuity payable until the last death among a group of lives.

Remark 3: In principle, and occasionally in practice, the group could consist of three or more lives. However, such policies are most commonly purchased by couples who are jointly organizing their financial security and we will restrict our attention to the case of two lives whom we will label, for convenience, ‘husband’ and ‘wife’.

Def. 4: A common benefit design is an annuity payable at a higher rate while both partners are alive and at a lower rate following the first death. The annuity ceases on the second death. This could be viewed as a last survivor annuity for the lower amount, plus a joint life annuity for the difference.

Def. 5: A **reversionary annuity** is a life annuity that starts payment on the death of a specified life, if his or her spouse is alive, and continues through the spouse's lifetime. A pension plan may offer a reversionary annuity benefit as part of the pension package, payable to the pension plan member's spouse for their remaining lifetime after the member's death.

Def. 6: Joint life insurance: a death benefit is paid on the first death of the husband and wife.

An appropriate model for these policies, the joint life and last survivor model, has four states:

‘Husband Alive, Wife Alive’ – State 0;
‘Husband Alive, Wife Dead’ – State 1;
‘Husband Dead, Wife Alive’ – State 2;
‘Husband Dead, Wife Dead’ – State 3;

Figure 8.5 The joint life and last survivor model.

Let x and y denote the ages of the husband and wife, respectively, when the annuity or insurance policy is purchased, at $t = 0$.

For $t \geq 0$,

$$Y(t) = \begin{cases} 0 & \text{if both husband and wife are alive at time } t \text{ (and aged } x+t \text{ and } y+t, \text{ respectively)} \\ 1 & \text{if the husband is alive at age } x+t \text{ and the wife died before age } y+t \\ 2 & \text{if the husband died before age } x+t \text{ and the wife is alive at age } y+t \\ 3 & \text{if both husband and wife are dead at time } t \end{cases}$$

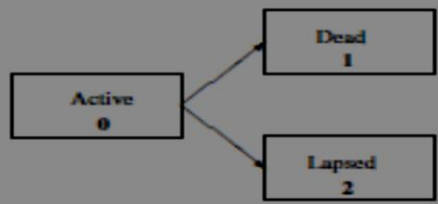


Figure 8.8 The insurance-with-lapses model.

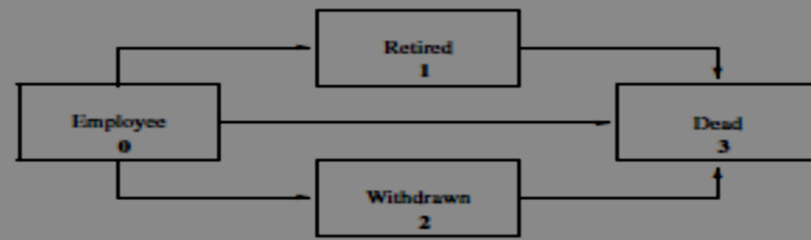
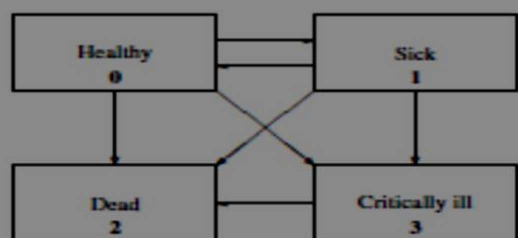


Figure 8.12 A withdrawal/retirement model.

Exercise 8.2 Consider the following model for an insurance policy combining disability income insurance benefits and critical illness benefits.



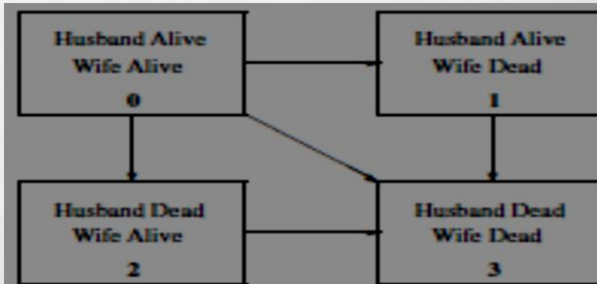


Figure 8.13 The common shock model.

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5.2 Assumptions and notation

Consider a general multiple state model with:

- a finite set of $n + 1$ states labelled $0, 1, \ldots, n$;
- instantaneous transitions being possible between selected pairs of states;
- these states represent different conditions for an individual or group of individuals;
- for each $t \geq 0$ define a random variable $Y(t)$ which takes one of the values $0, 1, \ldots, n$, and we interpret the event $Y(t) = i$ to mean that the group of lives being modelled is in state i at time t. The set of random variables $\{Y(t)\}_{t\geq 0}$ is then a continuous time stochastic process.

The multiple state model is an appropriate model for an insurance policy if the payment of benefits or premiums is dependent on being in a given state or moving between a given pair of states at a given time, as illustrated in the examples.

Remark that there is a natural starting state for the policy, always labelled state 0. This is the case for all examples based on multiple state models. For example, a policy providing an annuity during periods of sickness in return for premiums payable while healthy, would be issued only to a person who was healthy at that time.

Assumption 1.

For any states i and j and any times t and $t + s$, where $s \geq 0$, the conditional probability $Pr[Y(t+s) = j \mid Y(t) = i]$ does not depend on any information about the process before time t.

Intuitively, this means that the probabilities of future events for the process are completely determined by knowing its current state. In particular, these probabilities do not depend:

- on how the process arrived at the current state
- or how long it has been in the current state.

This property, that probabilities of future events depend on the present but not on the past, is known as the **Markov property**, i.e. we are assuming that $\{Y(t)\}_{t\geq 0}$ is a Markov process.

Assumption 1 is sometimes not made explicitly since it is unnecessary given the interpretation of the models in question. For instance, if we know that the process is in state 0 at time *x* (so that the individual is alive at age *x*) then we know the past of the process (the individual was alive at all ages before *x*).

Assumption 1 is more interesting, for example, in the disability income insurance model. If we know that $Y(t) = 1$, (the individual is sick at time t), then Assumption 1 says that the probability of any future move after time t (either recovery or death), does not depend on any further information, such as how long the life has been sick up to time t, or how many different periods of sickness the life has experienced up to time t. (In practice, we might believe that the probability of recovery in, say, the next week would depend on how long the current sickness has already lasted.)

Assumption 2.

For any positive interval of time h,

$$Pr[2 \text{ or more transitions within a time period of length } h] = o(h).$$

Recall that a function $g(h)$, is said to be $o(h)$ if $\lim_{h\to 0} \frac{g(h)}{h} = 0$ (as h converges to 0, $g(h)$ converges to zero faster than h.)

Assumption 2 tells us that for a small interval of time h, the probability of two or more transitions in that interval is so small that it can be ignored. This assumption is unnecessary for the models where only one transition can ever take place. However, it is a necessary assumption for technical reasons in the other situations. In these cases, given our interpretation of the models, it is not an unreasonable assumption.

Assumption 3.
For all states i and j and all ages $x \geq 0$, the probability that a life aged x in state i is in state j at age $x + t$, where j may be equal to i, is a differentiable function of t.

Assumption 3 is a technical assumption needed to ensure that the mathematics proceeds smoothly. A consequence of this assumption is that the probability of a transition taking place in a time interval of length t converges to 0 as t converges to 0.

These three assumptions are not too restrictive in practice.

Notation:

For states i and j in a multiple state model and for $x, t \geq 0$:

$${}_tp_x^{ij} = Pr[Y\,(x+t) = j \mid Y\,(x) = i] \quad (8.1)$$

$${}_tp_x^{\overline{ii}} = Pr[Y\,(x+s) = i \;\; \forall s \in [0,t] \mid Y\,(x) = i] \quad (8.2)$$

$$\mu_x^{ij} = \lim_{h \to 0^+} \frac{{}_hp_x^{ij}}{h} = \lim_{h \to 0^+} \frac{Pr[Y\,(x+h) = j \mid Y\,(x) = i]}{h}, i \neq j \qquad (8.3)$$

${}_tp_x^{ij}$ is the probability that a life aged x in state i is in state j at age $x + t$, where j may be equal to i;

${}_tp_x^{\overline{ii}}$ is the probability that a life aged x in state i stays in state i throughout the period from age x to age $x + t$;

μ_x^{ij} is the **force of transition** or **transition intensity** between states i and j, at age x.

Remark 7: Another consequence of Assumption 3 is that the limit in the definition of μ_x^{ij} always exists. It is also assumed that μ_x^{ij} is a bounded and integrable function of x.

Further, formula (8.3),

$$\mu_x^{ij} = \lim_{h \to 0^+} \frac{{}_h p_x^{ij}}{h} = \lim_{h \to 0^+} \frac{Pr[Y\,(x+h) = j \mid Y\,(x) = i]}{h}, i \neq j,$$

can be written

$${}_h p_x^{ij} = Pr[Y\,(x+h) \;=\; j \mid Y\,(x) \;=\; i] = h\mu_x^{ij} + o(h), \qquad h > 0 \quad (8.4)$$

Then, for small positive values of *h*

$${}_h p_x^{ij} \approx h\mu_x^{ij} \qquad (8.5)$$

Illustration 8: In terms of the alive–dead model the following observations follow:

$${}_tp_x^{00} = {}_tp_x$$
$${}_tp_x^{01} = {}_tq_x$$
$${}_tp_x^{10} = 0$$
$${}_0p_x^{ij} = 1, i = j$$
$${}_0p_x^{ij} = 0, i \neq j$$

$\mu_x^{01} = \mu_x$, the force of mortality at age x.

Formula (8.4):

$$h\mu_x^{01} \approx Pr[Y(x+h) = 1 \mid Y(x) = 0] = {}_hp_x^{01}$$

is equivalent to formula (2.8):

$$\mu_x dx \approx Pr[T_0 \leq x + dx \mid T_0 > x]$$

5.3 Transition intensities and probabilities

In Chapter 2, it was assumed that the force of mortality was known and formula ${}_tp_x = e^{-\int_x^{x+t}\mu_r dr} = e^{-\int_0^t \mu_{x+s} ds}$ was derived.

This same approach is adopted here where the transition intensities are known and formulae for all probabilities are derived in terms of them. $\left\{\mu_x^{ij}, x \geq 0, i, j = 0,1,\ldots,n, i \neq j\right\}$ are fundamental quantities which determine everything about a multiple state model.

Result 9:

For any state *i* in a multiple state model,

$$ {}_tp_x^{ii} = {}_tp_x^{\overline{ii}} + o(t) \qquad (8.6)$$

$$ {}_hp_x^{\overline{ii}} = 1 - h \sum_{j=0, j\neq i}^{n} \mu_x^{ij} + o(h), \qquad h > 0 \qquad (8.7)$$

$$ {}_tp_x^{\overline{ii}} = e^{-\int_0^t \sum_{j=0,j\neq i}^{n} \mu_{x+s}^{ij} ds} \qquad (8.8)$$

See Example 8.3.

Remark 10:

${}_{t+h}p_x^{\overline{ii}} = {}_tp_x^{\overline{ii}} \times {}_hp_{x+t}^{\overline{ii}}$ because ${}_{t+h}p_x^{\overline{ii}}$ is the probability that the individual/process stays in state *i* throughout the time period [0, *t* +*h*], given that the process was in state *i* at age *x* and this event can be split into two sub-events:

- the process stays in state *i* from age *x* until (at least) age *x* + *t*, given that it was in state *i* at age *x*,
- the process stays in state *i* from age *x* +*t* until (at least) age *x* +*t* +*h*, given that it was in state *i* at age *x* + *t*.

Further ${}_{t+h}p_x^{\overline{ii}} = {}_tp_x^{\overline{ii}} \times {}_hp_{x+t}^{\overline{ii}}$ can be written

$$ {}_{t+h}p_x^{\overline{ii}} = {}_tp_x^{\overline{ii}} \left(1 - h \sum_{j=0, j\neq i}^{n} \mu_{x+t}^{ij} + o(h)\right) $$

Remark 11:

The same reasoning can be applied to a general multiple state model to derive formulae for probabilities (Kolmogorov's forward equations).

To derive Kolmogorov's forward equations, we consider the probability of being in the required state, j, at age $x + t + h$, and condition on the state of the process at age $x + t$: either it is already in state j, or it is in some other state, say k, and a transition to j is required before age $x + t + h$.

Result 12:

Let i and j be any two, not necessarily distinct, states in a multiple state model which has a total of $n + 1$ states.

For $x, t \geq 0$, Kolmogorov's forward equations are:

$$\frac{d}{dt}\ {}_tp_x^{ij} = \sum_{k=0,k\neq j}^{n}\left({}_tp_x^{ik}\mu_{x+t}^{kj} - {}_tp_x^{ij}\mu_{x+t}^{jk}\right) \qquad (8.14)$$

$$\frac{d}{dt}\ {}_tp_x^{\overline{ii}} = -\ {}_tp_x^{\overline{ii}} \sum_{k=0,k\neq i}^{n} \mu_{x+t}^{ik} \Longrightarrow\ {}_tp_x^{\overline{ii}} = e^{-\int_0^t \sum_{k=0,k\neq i}^{n} \mu_{x+s}^{ik} ds} \qquad (8.8)$$

Particular cases

$n = 1$ (2 states)

$$\frac{d}{dt}\,{}_tp_x^{00} = {}_tp_x^{01}\mu_{x+t}^{10} - {}_tp_x^{00}\mu_{x+t}^{01};\ \frac{d}{dt}\,{}_tp_x^{01} = {}_tp_x^{00}\mu_{x+t}^{01} - {}_tp_x^{01}\mu_{x+t}^{10}$$

$$\frac{d}{dt}\,{}_tp_x^{10} = {}_tp_x^{11}\mu_{x+t}^{10} - {}_tp_x^{10}\mu_{x+t}^{01};\frac{d}{dt}\,{}_tp_x^{11} = {}_tp_x^{10}\mu_{x+t}^{01} - {}_tp_x^{11}\mu_{x+t}^{10}$$

$${}_tp_x^{\overline{00}} = e^{-\int_0^t \mu_{x+s}^{01}};\qquad {}_tp_x^{\overline{11}} = e^{-\int_0^t \mu_{x+s}^{10}}$$

$n = 2$ (3 states)

$$\frac{d}{dt}\,{}_tp_x^{00} = \left({}_tp_x^{01}\mu_{x+t}^{10} - {}_tp_x^{00}\mu_{x+t}^{01}\right) + \left({}_tp_x^{02}\mu_{x+t}^{20} - {}_tp_x^{00}\mu_{x+t}^{02}\right);$$

$$\frac{d}{dt}\,{}_tp_x^{01} = \left({}_tp_x^{00}\mu_{x+t}^{01} - {}_tp_x^{01}\mu_{x+t}^{10}\right) + \left({}_tp_x^{02}\mu_{x+t}^{21} - {}_tp_x^{01}\mu_{x+t}^{12}\right);\ \ldots$$

$${}_tp_x^{\overline{00}} = e^{-\int_0^t (\mu_{x+s}^{01}+\mu_{x+s}^{02})};\ldots$$

5.4 Numerical evaluation of probabilities

See Example 8.4

$$ {}_{t}p_{x}^{\overline{ii}} = e^{-\int_0^t \sum_{j=0, j\neq i}^{n} \mu_{x+s}^{ij} ds} \qquad (8.8)$$

$$ {}_{u}p_{x}^{01} = \int_0^u {}_{t}p_{x}^{\overline{00}} \mu_{x+t}^{01} \times {}_{u-t}p_{x+t}^{\overline{11}}\, dt \qquad (8.11)$$

and Example 8.5

$$ {}_{t+h}p_{x}^{ij} = {}_{t}p_{x}^{ij} - h \sum_{k=0, k\neq j}^{n} \left({}_{t}p_{x}^{ij} \mu_{x+t}^{jk} - {}_{t}p_{x}^{ik} \mu_{x+t}^{kj} \right) + o(h) \qquad (8.13)$$

SURVIVAL MODELS AND LIFE CONTINGENCIES

MASTER IN ACTUARIAL SCIENCE

LECTURE 21

5. MULTI-STATE POLICIES (Dickson *et al.* – Chap. 8, pp. 230-289)

5.5 Premiums

Multiple state models are a natural way of:

- modelling cash flows for insurance policies.
- evaluating probabilities for such models given the transition intensities between pairs of states.

To calculate premiums and policy values for a policy represented by MSM it is necessary to generalize the definitions of insurance and annuity functions to a multiple state framework.

Example 1:

Consider a life aged x currently in state i of a multiple state model. How to value an annuity of 1 per year payable continuously while the life is in some state j (which may be equal to i)?

The EPV of the annuity, at force of interest δ per year, is

$$\bar{a}_x^{ij} = E\left[\int_0^\infty e^{-\delta t} I(Y(t) = j | Y(0) = i) dt\right]$$

$$= \int_0^\infty e^{-\delta t} E[I(Y(t) = j | Y(0) = i)] dt = \int_0^\infty e^{-\delta t} \ {}_tp_x^{ij} dt,$$

I the indicator function.

Example (cont.):
How to value an annuity if it is payable at the start of each year, from the current time?

The EPV of the annuity is now

$$\ddot{a}_x^{ij} = \sum_{k=0}^{\infty} v^k \ {}_kp_x^{ij}$$

Annuity benefits payable more frequently can be valued similarly.

For insurance benefits, the payment is usually conditional on making a transition. A death benefit is payable on transition into the dead state; a critical illness insurance policy might pay a sum insured on death or earlier diagnosis of one of a specified group of illnesses.

Example 2

Suppose a unit benefit is payable immediately on each future transfer into state k, given that the life is currently in state i (which may be equal to k).

The EPV of the benefit is …

$$\bar{A}_x^{ik} = \int_0^\infty \sum_{j \neq k} e^{-\delta t} \ {}_tp_x^{ij} \mu_{x+t}^{jk} \, dt \qquad (8.16)$$

As usual, to derive this EPV, consider payment in the interval *(t, t + dt)* and that:

- the amount of the payment is 1;
- the discount factor is $e^{-\delta t}$;
- the probability that the benefit is paid is the probability that the life transfers into state *k* in *(t, t + dt)*, given that the life is in state *i* at time 0. In order to transfer into state *k* in *(t, t + dt)*, the life must be in some state *j* that is not *k* immediately before (the probability of two transitions in infinitesimal time being negligible), with probability ${}_tp_x^{ij}$, then transfer from *j* to *k* during the interval *(t, t + dt)*, with probability (loosely) $\mu_{x+t}^{jk} dt$.

Summing (that is, integrating) over all the possible time intervals gives equation (8.16).

Remark the differences between the two previous cases, where the three key points listed above show clearly:

The value of an annuity of 1 per year payable at the start of each year to a life aged *x*, currently in state *i*, while the life is in some state *j* is

$$\ddot{a}_x^{ij} = \sum_{k=0}^{\infty} 1 v^k \ {}_k p_x^{ij}$$

The value of a unit benefit payable immediately on each future transfer into state *k*, given that the life is currently in state *i* is

$$\bar{A}_x^{ik} = \int_0^{\infty} \sum_{j \neq k} 1 e^{-\delta t} \ {}_t p_x^{ij} \mu_{x+t}^{jk} \, dt$$

Other benefits and annuity designs are feasible. Most practical cases can be managed from **first principles** using the indicator function approach.

In general, premiums are calculated using the equivalence principle, assuming that lives are in state 0 at the policy inception date.

5.6 Policy values and Thiele's differential equation generalisation

The definition of the time *t* policy value for a policy modeled using a multiple state model is as before – it is the expected value at that time of the future loss random variable – with one additional requirement.

For a policy described by a multiple state model, the future loss random variable, and hence the policy value, at duration *t* years depends on which state of the model the policyholder is in at that time.

A policy value is then the expected value at that time t of the future loss random variable *conditional* on the policy being in a given state *i* at that time. It is denoted by ${}_tV^{(i)}$.

Consider an insurance policy issued at age x and with term n years described by a multiple state model with $n+1$ states, labelled $0,1,\ldots,n$.

Notation:

μ_y^{ij}: transition intensities between states i and j at age y.

δ_t: force of interest per year at time t.

$B_t^{(i)}$: rate of payment of benefit while the policyholder is in state i. The premiums are included as negative benefits and expenses are included as additions to the benefits

$S_t^{(ij)}$: lump sum benefit payable instantaneously at time t on transition from state i to state j.

$\delta_t, B_t^{(i)}$ and $S_t^{(ij)}$ are continuous functions of t.

For this very general model, $i = 0,1, \ldots, n$ and $0 \leq t \leq n$, Thiele's differential equation, which can be solved numerically using Euler's method, is

$$\frac{d}{dt}\,{}_tV^{(i)} = \delta_t\,{}_tV^{(i)} - B_t^{(i)} - \sum_{j=0,j\neq i}^{n} \mu_{x+t}^{ij}\left(S_t^{(ij)} + {}_tV^{(j)} - {}_tV^{(i)}\right)$$

(8.23)

5.7 Multiple decrement models (competing risks)

Multiple decrement models are special types of multiple state models which occur frequently in actuarial applications.

A multiple decrement model is characterized by having a single starting state and several exit states with a possible transition from the starting state to any of the exit states, but no further transitions. The accidental death model is an example of such a model with two exit states.

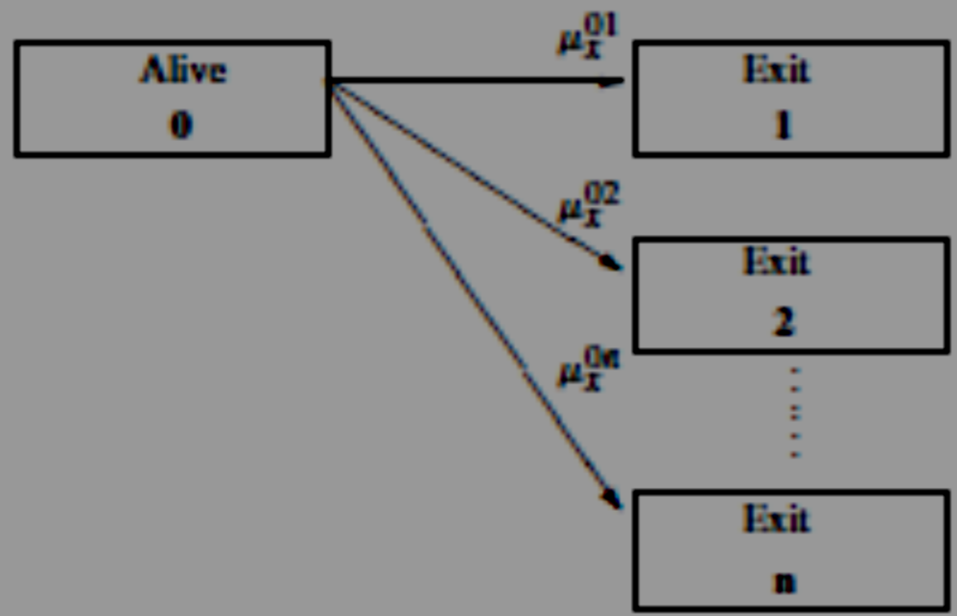

Figure 8.7 A general multiple decrement model.

Calculating probabilities for a multiple decrement model is relatively easy since only one transition can ever take place. For such a model we have for $i = 1,2,\dots,n$ and $j = 0,1,\dots,n$ $(j \neq i)$,

$$ {}_tp_x^{00} = {}_tp_x^{\overline{00}} = e^{-\int_0^t \sum_{i=1}^n \mu_{x+s}^{0i} ds}, $$

$$ {}_tp_x^{0i} = \int_0^t {}_sp_x^{00} \mu_{x+s}^{0i} ds, $$

$$ {}_0p_x^{ii} = 1, $$

$$ {}_0p_x^{ij} = 0. $$

Assuming the transition intensities as functions of x are known, it is possible to calculate ${}_tp_x^{00}$ and ${}_tp_x^{0i}$ using numerical or analytic integration.

MULTI-STATE POLICIES (Dickson *et al.* – Chap. 8, pp. 230-289)
5.8 Joint life status and last survivor status

(Institute and Faculty of Actuaries, *Subject CT5- Contingencies - Core Technical - Core Reading*
UNIT 6 - ANNUITIES AND ASSURANCES INVOLVING TWO LIVES)

5.8.1 The model and assumptions

Consider the valuation of benefits and premiums for an insurance policy where payments depend on the survival or death of two lives, husband' and 'wife'. Such policies are very common. Policies relating to three or more lives also exist, but are far less common.

Consider future payments from a time when both husband and wife are alive and are aged x and y, respectively. It is necessary to evaluate probabilities of survival/death for the two lives using an adequate model. Consider the model below.

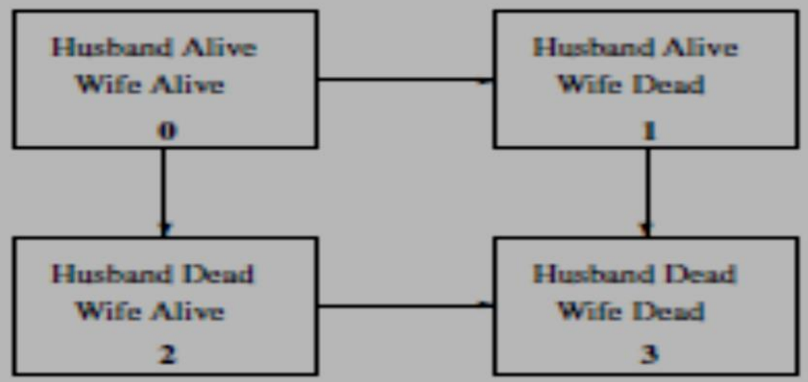

Figure 8.5 The joint life and last survivor model.

Notation:

$\mu^{01}_{x+t:y+t}$ is the intensity of mortality for the wife when she is aged $y+t$ given that her husband is still alive and that he is aged $x+t$.

…

${}_tp^{13}_{x+u}$ denotes the probability that the husband, who is now aged $x+u$ and whose wife has already died, dies before reaching age $x+u+t$. The exact age at which the wife died is assumed to be irrelevant and so is not part of the notation.

Although this is a death probability, the multi state notation makes no use of the letter q. The standard actuarial notation for joint life benefits differs from the general multiple state model notation.

…

	Actuarial notation	Multi state notation
Pr[(x) and (y) are both alive in t years]	${}_tp_{xy}$	${}_tp_{xy}^{00}$
Pr[(x) and (y) are not both alive in t years]	${}_tq_{xy}$	${}_tp_{xy}^{01} + {}_tp_{xy}^{02} + {}_tp_{xy}^{03}$
Pr[(x) dies before $(y)$and within t years]	${}_tq_{xy}^1$	$\int_0^t {}_rp_{xy}^{00}\, \mu_{x+r:y+r}^{02} dr$
Pr[(x) dies after $(y)$and within t years]	${}_tq_{xy}^2$	$\int_0^t {}_rp_{xy}^{01}\, \mu_{x+r}^{13} dr$
Pr[at least one of $(x)$and (y) is alive in t years]	${}_tp_{\overline{xy}}$	${}_tp_{xy}^{00} + {}_tp_{xy}^{01} + {}_tp_{xy}^{02}$
Pr[(x) and (y) are both dead in t years]	${}_tq_{\overline{xy}}$	${}_tp_{xy}^{03}$

The subscript xy or $\overline{xy}$ is the **status**. The q-type probabilities are associated with the *failure* of the status – the joint life status xy fails on the first death of (x) and (y) , and the last survivor status $\overline{xy}$ fails on the last death of (x) and (y) .

The random variables of interest are T_x and T_y, the future lifetimes of two lives, one aged x and the other aged y.

So far we have described annuity and assurance functions which depend upon the death or survival of a single life aged *x*.

We now consider annuity and assurance functions which depend upon the death or survival of two lives.

The random variables of interest are T_x and T_y, the future lifetimes of the two lives, one aged *x* and the other aged *y*.

Assumption:
T_x **and** T_y **are independent random variables.**

5.8.2 Joint life functions

Def.14

The random variable T_{xy} measures the joint lifetime of (x) and (y) i.e. the time while both (x) and (y) remain alive, which is the time until the first death of (x) and (y), that is

$T_{xy} = min\{T_x, T_y\}$

Def. 15

The cumulative distribution function of T_{xy} is

$$F_{T_{xy}}(t) = Pr\left(T_{xy} \leq t\right).$$

Exercise:

Prove that $F_{T_{xy}}(t) = 1 - {}_tp_x\,{}_tp_y$

and that the density function of T_{xy} is $f_{T_{xy}}(t) = {}_tp_x\,{}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right)$

(Hint: recall that $\frac{d}{dt}\,{}_tp_x = -{}_tp_x\mu_{x+t}$)

Life table functions are an aid to find the solutions of actuarial problems involving a single life.

In a similar way it is helpful to develop the joint life functions to help in the numerical evaluation of expressions which are the solution to problems involving more than one life.

$$ {}_tp_{xy} = {}_tp_x \, {}_tp_y = \frac{l_{x+t}}{l_x}\frac{l_{y+t}}{l_y} = \frac{l_{x+t:y+t}}{l_{xy}}, l_{x+t:y+t} = l_{x+t}l_{y+t};\ l_{xy} = l_x l_y $$

$$ d_{xy} = l_{xy} - l_{x+1:y+1} $$

$$ q_{xy} = \frac{d_{xy}}{l_{xy}} $$

The force of failure of the joint life status can be derived in the usual way

$$\mu_{x+t:y+t} = -\frac{\frac{d\left(l_{x+t:y+t}\right)}{dt}}{l_{x+t:y+t}} = -\frac{\frac{d\left({}_t p_{xy}\right)}{dt}}{{}_t p_{xy}}$$

Exercise:

Prove that $\mu_{x+t:y+t} = \mu_{x+t} + \mu_{y+t}$.

Remark 16: this relationship is additive in contrast to the previous relationships which were multiplicative, and that there is no “simple” relationship for d_{xy}.

Further,

$$f_{T_{xy}}(t) = {}_tp_x\,{}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right) = {}_tp_{xy}\mu_{x+t:y+t}.$$

Def. 17:

The discrete random variable which measures the curtate joint future lifetime of (x) and (y) is $K_{xy} =$ integer part of T_{xy}.

Exercise:

Justify that the probability function of K_{xy} is $Pr(K_{xy} = k) = Pr(k \leq T_{xy} < k + 1) = {}_k p_{xy} - {}_{k+1} p_{xy}$

Justify that the probability function of K_{xy} is also given by

$$Pr(K_{xy} = k) = {}_{k|} q_{xy} = {}_k p_{xy} \times q_{x+k:y+k}.$$

Then

$$Pr(K_{xy} \leq k) = Pr(\text{integer part of } T_{xy} \leq k)$$

$$= Pr(\text{integer part of } min\{T_x, T_y\} \leq k)$$

$$= \sum_{t=0}^{k} {}_t p_x \, {}_t p_y (1 - p_{x+t} p_{y+t})$$

Remark 18:

The joint life table functions l_{xy}, d_{xy}, q_{xy} and μ_{xy} are not tabulated in the “Formulae and Tables for Examinations”. However, these functions can be evaluated using the tabulated single life functions.

Example:
Question 8.6 (CR CT5)

Question 8.6

Assuming the mortality of AM92 (Ultimate) for both lives, calculate the following:

(i) ${}_3p_{45:41}$

(ii) $q_{66:65}$

(iii) $\mu_{38:30}$

Solution 8.6

(i) $${}_3p_{45:41}=\frac{l_{48:44}}{l_{45:41}}=\frac{l_{48}l_{44}}{l_{45}l_{41}}=\frac{9{,}753.4714\times 9{,}814.3359}{9{,}801.3123\times 9{,}847.0510}=0.9918$$

(ii) $$q_{66:65}=1-p_{66:65}=1-\frac{l_{67:66}}{l_{66:65}}=1-\frac{l_{67}}{l_{66}}\frac{l_{66}}{l_{65}}=1-\frac{l_{67}}{l_{65}}=0.030$$

(iii) $$\mu_{38:30}=\mu_{38}+\mu_{30}=0.000788+0.000585=0.001373$$

SURVIVAL MODELS AND LIFE CONTINGENCIES

MASTER IN ACTUARIAL SCIENCE

LECTURE 22

5.8.3 Last survivor functions

Def. 19

The random variable $T_{\overline{xy}}$ measures the time until the last death of (*x*) and (*y*) i.e. the time while at least one of (*x*) and (*y*) remains alive, that is $T_{\overline{xy}} = max\{T_x, T_y\}$

Def. 20

The cumulative distribution function of $T_{\overline{xy}}$ is

$$F_{T_{\overline{xy}}}(t) = Pr\left(T_{\overline{xy}} \leq t\right).$$

Exercise:

Prove that $F_{T_{\overline{xy}}}(t) = F_{T_x}(t) + F_{T_y}(t) - F_{T_{xy}}(t)$ and that the density function of $T_{\overline{xy}}$ is $f_{T_{\overline{xy}}}(t) = f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t)$

$= {}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_x\,{}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right),$

${}_tp_x = S_x(t) = e^{-\int_x^{x+t} \mu_r dr} = e^{-\int_0^t \mu_{x+s} ds} \ldots$

$$F_{T_{\overline{xy}}}(t) = Pr\left(T_{\overline{xy}} \leq t\right)$$

$$= P\left(\max\{T_x, T_y\} \leq \mathrm{t}\right) = P\left(T_x \leq t, T_y \leq t\right) = P(T_x \leq t)P\left(T_y \leq t\right)$$

$$= {}_tq_x \ {}_tq_y = \left(1 - {}_tp_x\right)\left(1 - {}_tp_y\right)$$

$$= 1 - {}_tp_x - {}_tp_y + {}_tp_x \ {}_tp_y$$

$$= \left(1 - {}_tp_x\right) + \left(\mathbf{1} - {}_tp_y\right) - \left(\mathbf{1} - {}_tp_x \ {}_tp_y\right)$$

$$= F_{T_x}(t) + F_{T_y}(t) - F_{T_{xy}}(t)$$

Def. 21:

The discrete random variable which measures the curtate last survivor lifetime of (x) and (y) is

$$K_{\overline{xy}} = \text{integer part of } T_{\overline{xy}}$$

Exercise:

Prove that the probability function of $K_{\overline{xy}}$ is

$$Pr\left(K_{\overline{xy}} = k\right) = Pr\left(k \leq T_{\overline{xy}} < k+1\right) = {}_{k|}q_{\overline{xy}} = {}_{k|}q_x + {}_{k|}q_y - {}_{k|}q_{xy}.$$

Remark 22:

- The cumulative distribution function of $T_{\overline{xy}}$ is
$$P\left(T_{\overline{xy}} \leq t\right) = 1 - {}_tp_x - {}_tp_y + {}_tp_x\ {}_tp_y$$
- The survival function is
$$P\left(T_{\overline{xy}} > t\right) = {}_tp_x + {}_tp_y - {}_tp_x\ {}_tp_y$$
- The survival function can be factorised into
$${}_tp_x\ {}_tp_y + \left(1 - {}_tp_x\right)\ {}_tp_y + \left(1 - {}_tp_y\right)\ {}_tp_x$$

where each of the three terms corresponds to one of the mutually exclusive and exhaustive events which result in the last survivor of (x) and (y) living for at least t years i.e.

  - both (x) and (y) alive after t years
  - (x) dead, but (y) alive after t years
  - (x) alive, but (y) dead after t years

These probabilities can be evaluated directly.

- The probability of the complementary event, that both lives will die within t years, is $P\left(T_{\overline{xy}} \leq t\right) = \left(1 - {}_tp_x\right)\left(1 - {}_tp_y\right)$

Remark 23:

It has already been derived that

$$F_{T_{\overline{xy}}}(t) = F_{T_x}(t) + F_{T_y}(t) - F_{T_{xy}}(t) = 1 - {}_tp_x - {}_tp_y + {}_tp_x\ {}_tp_y$$

and that

$$f_{T_{\overline{xy}}}(t) = f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t)$$
$$= {}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t}$$

and so it seems that all last survivor functions can be expressed in terms of single life and joint life functions. This is true and provides a method of evaluating such functions without the need to develop any additional functions to help in computation.

Example: Question 8.7 (CR CT5)

Question 8.7

Calculate $p_{\overline{62:65}}$ and ${}_3q_{\overline{50:50}}$ using AM92 Ultimate mortality.

Solution 8.7

(i) $p_{\overline{62:65}} = 1 - q_{\overline{62:65}} = 1 - q_{62}q_{65} = 1 - 0.010112 \times 0.014243 = 0.99986$

or:

$$p_{\overline{62:65}} = p_{62} + p_{65} - p_{62:65} = \frac{l_{63}}{l_{62}} + \frac{l_{66}}{l_{65}} - \frac{l_{63:66}}{l_{62:65}} = 0.99986$$

(ii) $_3q_{\overline{50:50}} = \left({}_3q_{50}\right)^2 = \left(1 - \frac{l_{53}}{l_{50}}\right)^2 = \left(1 - \frac{9,630.0522}{9,712.0728}\right)^2 = 0.0000713$

Note that for joint life statuses it is often easier to work with *p*-type functions because

$$ {}_tp_{xy} = {}_tp_x \times {}_tp_y. $$

For last survivor statuses is is often easier to work with *q*-type functions because

$$ {}_tq_{\overline{xy}} = {}_tq_x \times {}_tq_y. $$

This is the result of the relationship between the joint lifetime and last survivor lifetime random variables:

$T_{xy} + T_{\overline{xy}} = min\ \{T_x\ , T_y\} + max\{T_x\ , T_y\} = T_x + T_y$
$K_{xy} + K_{\overline{xy}} = min\ \{K_x\ , K_y\} + max\{K_x\ , K_y\} = K_x + K_y$

This last equation gives the result $K_{\overline{xy}} = K_x + K_y - K_{xy}$ and

$$Pr\left(K_{\overline{xy}} = k\right) = Pr(K_x = k) + Pr\left(K_y = k\right) - Pr\left(K_{xy} = k\right),$$

another way to write

$$Pr\left(K_{\overline{xy}} = k\right) = {}_{k|}q_{\overline{xy}} = {}_{k|}q_x + {}_{k|}q_y - {}_{k|}q_{xy}.$$

So curtate last survivor functions can be evaluated from the corresponding **joint** life and **single** life functions.

5.8.4 Present values of joint life and last survivor assurances

Consider an assurance under which the benefit (of 1) is paid immediately on the ending (failure) of a status u. This status u could be any joint lifetime or last survivor status, xy or $\overline{xy}$. Let T_u be a (continuous) random variable representing the future lifetime of the status u and let $f_{T_u}(t)$ be the probability density function of T_u.

The present value of the assurance can be represented by the random variable

$$\bar{Z}_u = v_i^{T_u}$$

where i is the valuation rate of interest.

The expected value of $\bar{Z}_u$ is denoted by $\bar{A}_u$,

$$\bar{A}_u = E[\bar{Z}_u] = \int_0^{\infty} v_i^t \, f_{T_u}(t)dt$$

And the variance can be written as

$$Var(\bar{Z}_u) = E[\bar{Z}_u^2] - (E[\bar{Z}_u])^2 = \int_0^{\infty} v^{2t} \, f_{T_u}(t)dt - (\bar{A}_u)^2$$

$$= {}^2\bar{A}_u - (\bar{A}_u)^2$$

Where ${}^2\bar{A}_u$ is evaluated at a valuation rate of interest of $i^* = 2i + i^2$.

If the assurance benefit is paid at the end of the year in which the status finishes, then a (discrete) random variable K_u can be used, with a present value function. The present value of the assurance can be represented by the random variable

$$Z_u = v_i^{K_u+1},$$

where i is the valuation rate of interest.

5.8.4.1 present values of joint life assurances

When $u = xy$, the mean and variance of the present value of an assurance payable immediately on the failure of the joint lifetime of (*x*) and (*y*) are

$$\bar{A}_{xy} = E[\bar{Z}_{xy}] = \int_0^\infty v_i^t \underbrace{{}_tp_x \ {}_tp_y}_{{}_tp_{xy}} \underbrace{(\mu_{x+t} + \mu_{y+t})}_{\mu_{x+t:y+t}} dt$$

$$Var(\bar{Z}_{xy}) = E[\bar{Z}_{xy}^2] - (E[\bar{Z}_{xy}])^2 = {}^2\bar{A}_{xy} - (\bar{A}_{xy})^2$$

where ${}^2\bar{A}_{xy}$ is evaluated at a valuation rate of interest of $i^* = 2i + i^2$.

A similar analysis for K_{xy} gives the mean and variance of the present value of the joint life assurance with sum assured payable at the end of the year of failure.

$$A_{xy} = E[Z_{xy}] = \sum_{k=0}^{\infty} v_i^{k+1} \, {}_{k|}q_{xy}$$

$$Var(Z_{xy}) = E[Z_{xy}^2] - (E[Z_{xy}])^2 = {}^2A_{xy} - (A_{xy})^2$$

5.8.4.2 present values of last survivor assurances

When $u = \overline{xy}$, the mean and variance of the present value of an assurance payable immediately on the death of the last survivor of (*x*) or (*y*) are

$$\bar{A}_{\overline{xy}} = E\left[\bar{Z}_{\overline{xy}}\right] = \int_0^\infty v_i^t \left({}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t}\right)dt$$

$$= \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$$

$$Var\left(\bar{Z}_{\overline{xy}}\right) = E\left[\bar{Z}^2_{\overline{xy}}\right] - \left(E\left[\bar{Z}_{\overline{xy}}\right]\right)^2$$

$$= \left({}^2\bar{A}_x + {}^2\bar{A}_y - {}^2\bar{A}_{xy}\right) - \left(\bar{A}_x + \bar{A}_y - \bar{A}_{xy}\right)^2$$

Where ${}^2\bar{A}_{xy}$ is evaluated at a valuation rate of interest of $i^* = 2i + i^2$. Again, last survivor functions can be evaluated in terms of single and joint life functions.

A similar analysis for $K_{\overline{xy}}$ gives the mean and variance of the present value of the joint life assurance with sum assured payable at the end of the year of failure.

$$A_{\overline{xy}} = E\left[Z_{\overline{xy}}\right] = A_x + A_y - A_{xy}$$

$$Var\left(Z_{\overline{xy}}\right) = E\left[Z_{\overline{xy}}^2\right] - \left(E\left[Z_{\overline{xy}}\right]\right)^2 = {}^2A_{\overline{xy}} - \left(A_{\overline{xy}}\right)^2 = \left({}^2A_x + {}^2A_y - {}^2A_{xy}\right) - \left(A_x + A_y - A_{xy}\right)^2$$

5.8.5 present values of joint life and last survivor annuities

Consider an **annuity** under which the benefit (of 1) is **paid continuously** so long as a status u continues. This status u could be any joint lifetime or last survivor status, xy or $\overline{xy}$.

The present value of these annuity payments can be represented by the random variable $\bar{a}_{\overline{T_u}|}$ with expected value (EPV)

$$\bar{a}_u = E\left[\bar{a}_{\overline{T_u}|}\right] = \int_0^\infty \bar{a}_{\overline{t}|}\, f_{T_u}(t)dt = E\left[\frac{1 - v^{T_u}}{\delta}\right] = \frac{1 - E[v^{T_u}]}{\delta}$$

$$= \frac{1 - \bar{A}_u}{\delta}$$

The variance can be expressed in a similar way

$$Var\left(\bar{a}_{\overline{T_u}|}\right) = Var\left(\frac{1-v^{T_u}}{\delta}\right) = \frac{1}{\delta^2}Var(v^{T_u}) = \frac{1}{\delta^2}\left({}^2\bar{A}_u - (\bar{A}_u)^2\right).$$

For instance,

$$\bar{a}_{\overline{xy}} = \bar{a}_x + \bar{a}_y - \bar{a}_{xy}.$$

The payment stream for the last survivor annuity is equivalent to continuous payments at unit rate per year to both husband and wife while each of them is alive minus a continuous payment at unit rate per year while both are alive. This gives a net payment at unit rate per year while at least one of them is alive, which is what we want.

	In advance	In arrear
Random Variable (discrete)	$\ddot{a}_{\overline{K_u+1}\vert}$	$a_{\overline{K_u}\vert}$
Mean	$\ddot{a}_u = \dfrac{1 - A_u}{d}$	$a_u = \dfrac{(1-d) - A_u}{d}$
Variance	$\dfrac{1}{d^2}\left({}^2A_u - (A_u)^2 \right)$	$\dfrac{1}{d^2}\left({}^2A_u - (A_u)^2 \right)$

The means and variances of the present values of annuities **payable in advance and in arrear** can be evaluated according to the table above.

These results can be used to determine the means and variances for $u=xy$ (the joint life annuity) and $u=\overline{xy}$ (the last survivor annuity).

5.8.6 EPV of joint life and last survivor assurances and annuities which depend upon term

5.8.6.1 EPV of joint life assurances and annuities which also depend upon term

Joint life assurances which are dependent on a fixed term of *n* years can be **term assurances** or **endowment assurances**.

Their expected present values if they are paid immediately on death can be expressed as:

$$\bar{A}^{1}_{\overset{\frown}{xy}:\overline{n|}} = \int_0^n v^t \; {}_tp_{xy}\mu_{x+t:y+t}\,dt \qquad \text{(term assurance)}$$

$$\bar{A}_{xy:\overline{n|}} = \bar{A}^{1}_{\overset{\frown}{xy}:\overline{n|}} + A_{xy:\overset{1}{\overline{n|}}}, \quad A_{xy:\overset{1}{\overline{n|}}} = {}_np_{xy}v^n \; \text{(endowment assurance)}$$

The bracket used in the notation for the term assurance indicates that the joint status must end within the fixed term of *n* years for the benefit to be payable.

The expected present value of the temporary joint life annuity payable continuously can be written as

$$\bar{a}_{xy:\overline{n|}} = \int_0^n v^t \ {}_tp_{xy} \, dt$$

Similar expressions involving summation operators can be developed if assurances are paid at the end of the year of death or if annuities are payable annually in advance or in arrear.

5.8.6.2 EPV of last survivor assurances and annuities which also depend upon term

Last survivor assurances which are dependent on a fixed term of *n* years can be **term assurances** or **endowment assurances**. Their expected present values can be expressed in terms of single and joint life functions by making use of the results set out before.

$$\bar{A}^{\;1}_{\overline{xy}:\overline{n|}} = \bar{A}^{\;1}_{x:\overline{n|}} + \bar{A}^{\;1}_{y:\overline{n|}} - \bar{A}^{\;1}_{\overset{\frown}{xy}:\overline{n|}}$$

$$\bar{A}_{\overline{xy}:\overline{n|}} = \bar{A}_{x:\overline{n|}} + \bar{A}_{y:\overline{n|}} - \bar{A}_{xy:\overline{n|}}$$

The expected present value of the temporary last survivor annuity payable continuously can be written as

$$\bar{a}_{\overline{xy}:\overline{n|}} = \bar{a}_{x:\overline{n|}} + \bar{a}_{y:\overline{n|}} - \bar{a}_{xy:\overline{n|}}$$

Similar expressions involving summation operators can be developed if assurances are paid out at the end of the year of death or if annuities are payable annually in advance or in arrear.

5.8.7 EPV of joint life and last survivor annuities payable *m* times a year

For a single life status *x* the approximation

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m}$$

has been derived. Then

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m} \approx \underbrace{\ddot{a}_x}_{1+a_x} - \frac{m-1}{2m} - \frac{1}{m}, \qquad \text{i.e.}$$

$$a_x^{(m)} \approx a_x + \frac{m-1}{2m}$$

It is important to note that the nature of the above approximation means that the single life status *x* can equally be replaced by any life status, "*u*" say.

$$a_{xy}^{(m)} \approx a_{xy} + \frac{m-1}{2m}$$

$$a_{\overline{xy}}^{(m)} \approx a_x + a_y - a_{xy} + \frac{m-1}{2m}$$

$$a_{xy:\overline{n|}}^{(m)} \approx a_{xy:\overline{n|}} + \frac{m-1}{2m}\left(1 - {}_np_{xy}v^n\right)$$

$$a_{\overline{xy}:\overline{n|}}^{(m)}$$
$$\approx a_{x:\overline{n|}} + a_{y:\overline{n|}} - a_{xy:\overline{n|}}$$
$$+ \frac{m-1}{2m}\left[1 - \left({}_np_x\, v^n + {}_np_y\, v^n - {}_np_{xy}v^n\right)\right]$$

SURVIVAL MODELS AND LIFE CONTINGENCIES

MASTER IN ACTUARIAL SCIENCE

LECTURE 23

1. Characterise the policy with EPV $88\,000\bar{A}_{\overline{50:50}}$ and calculate this EPV given that $\mu = 0.04$ throughout the first life and $\mu = 0.03$ throughout the second life. The force of interest is $\delta = 5\%$ per annum.

Solution

$88\,000\bar{A}_{\overline{50:50}}$ is the EPV of an assurance paying 88 000 immediately on the death of the last survivor of two independent lives currently aged 50.

$$\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$$

$$\bar{A}_x = \int_0^\infty v^t\, {}_tp_x \mu_{x+t}\, dt;\ \bar{A}_y = \int_0^\infty v^t\, {}_tp_y \mu_{y+t}\, dt;\ \bar{A}_{xy} = \int_0^\infty v^t\, \underbrace{{}_tp_x\, {}_tp_y}_{{}_tp_{xy}} \underbrace{(\mu_{x+t} + \mu_{y+t})}_{\mu_{x+t:y+t}}\, dt$$

$$\bar{A}_{50} = \int_0^\infty e^{-\int_0^t 0.05dr} e^{-\int_0^t 0.04ds} (0.04)dt = 0.(4)$$

$$\bar{A}_{50} = \int_0^\infty e^{-\int_0^t 0.05dr} e^{-\int_0^t 0.03ds} (0.03)dt = 0.375$$

$$\bar{A}_{50:50} = \int_0^\infty e^{-\int_0^t 0.05dr} e^{-\int_0^t 0.07ds} (0.07)dt = 0.58(3)$$

$$88\,000\bar{A}_{\overline{50:50}} = 88\,000\big(0.(4) + 0.375 - 0.58(3)\big) = 20\,777.(7)$$

2. A single premium annuity policy is now issued to a male aged exactly 60 and a woman aged exactly 50. The annuity of 30 000 payable annually in arrear commences immediately and continues while at least one of the lives is alive.

a) Calculate the single premium using the following basis:

> Mortality: PMA92C20 for the male life and PFA92C20 for the female life. The lives are independent with respect to mortality.
>
> Interest: 4% per annum.
>
> Expenses: initial expenses of 150; annuity expenses of 1% of the annuity payment, while the annuity is being paid.

b) Calculate the probability that the insurance company makes a profit on the contract given that it sold the annuity for 600 000.

Solution

a)

$$P = 150 + 1.01 \times 30\,000 \mathrm{a}_{\overline{60:50}} \approx 150 + 30\,300(\mathrm{a}_{60} + \mathrm{a}_{50} - \mathrm{a}_{60:50}) =$$
$$150 + 30\,300(14.632 + 18.539 - 14.161) = 576\,153.$$

b)

$$P(Profit) = P\left(150 + 30\,300 a_{\overline{K_{\overline{60:50}}}|} < 600\,000\right) = P\left(a_{\overline{K_{\overline{60:50}}}|} < 19.797\right)$$
$$= P\left(\frac{1 - v^{K_{\overline{60:50}}}}{0.04} < 19.797\right)$$
$$= P\left(\frac{1 - v^{K_{\overline{60:50}}}}{0.04} < 19.797\right) = P\left(v^{K_{\overline{60:50}}} > 0.20812\right) = P\left(K_{\overline{60:50}} \underbrace{\log v}_{<0} > \log 0.20812\right)$$
$$= P\left(K_{\overline{60:50}} < \frac{\log 0.20812}{\log \frac{1}{1.04}}\right) = P\left(K_{\overline{60:50}} < \frac{\log 0.20812}{\log \frac{1}{1.04}}\right) = P\left(K_{\overline{60:50}} < 40.02\right) = P\left(K_{\overline{60:50}} < 41\right)$$
$$= ({}_{41}q_{60})({}_{41}q_{50}) = \left(1 - \frac{l_{101}}{l_{60}}\right) \times \left(1 - \frac{l_{91}}{l_{50}}\right)$$
$$= \left(1 - \frac{171\,171}{9\,826\,131}\right) \times \left(1 - \frac{3\,706\,149}{9\,947\,452}\right)$$
$$= 0.6165.$$

3. A male life aged 58 exact and a female life aged 55 exact take out a whole life assurance policy. The policy pays a sum assured of 150 000 immediately on first death. Premiums are payable for a maximum period of seven years, half-yearly in advance, ceasing on first death. Calculate the half-yearly premium payable. Note that $A_{xy} = 1 - d\ddot{a}_{xy}$ and ${}_{7}E_{58:55} = 0.73470$.

Basis:

Mortality: PMA92C20 (male life), PFA92C20 (female life)

Rate of interest: 4% per annum

Expenses: Nil

Solution

$P =$ half-yearly premium

$$150\,000\bar{A}_{58:55} - 2P\ddot{a}^{(2)}_{58:55:\overline{7|}} = 0 \Rightarrow P = \frac{150\,000\bar{A}_{58:55}}{2\ddot{a}^{(2)}_{58:55:\overline{7|}}}$$

$$A_{58:55} = 1 - d\ddot{a}_{58:55} = 1 - \frac{0.04}{1.04}15.306 = 0.41131$$

$$\ddot{a}^{(2)}_{58:55:\overline{7|}} = \ddot{a}^{(2)}_{58:55} - {}_7E_{58:55}\ddot{a}^{(2)}_{65:62} = \left(\ddot{a}_{58:55} - \frac{1}{4}\right) - 0.7347\left(\underbrace{\ddot{a}_{65:62}}_{12.427} - \frac{1}{4}\right)$$

$$= 6.1096$$

Then, $P = \frac{150\,000 \times 1.04^{\frac{1}{2}} \times 0.41131}{2 \times 6.1096} = 5149.173$

5. MULTI-STATE POLICIES (Dickson *et al.* – Chap. 8, pp. 230-289)

5.8 Joint life status and last survivor status

(…)

Contingent assurances – which are payable on the death of one life, contingent upon another life being in a specified state (alive or dead)

Reversionary annuities – which are payable to one life from the moment of death of another life.

See also pp. 162 and ff CM1 Core Reading (2019 ed.)

5.8.8 EPV of contingent assurances and reversionary annuities

5.8.8.1Contingent probabilities of death

So far events have been studied which depend on the joint lifetime xy or last survivor lifetime $\overline{xy}$ of two lives. One can also look at events which depend on the order in which the deaths occur:

$\overset{1}{xy}$: the event that (*x*) is the first to die of two lives (*x*) and (*y*).

$\overset{2}{xy}$: the event that (*x*) is the second to die of two lives (*x*) and (*y*).

Events which depend upon the order in which the lives die are called contingent events.

${}_nq_{xy}^1$ and ${}_nq_{xy}^2$ are used to denote that probabilities that each of these two events occurs in the next n years. These probabilities can be evaluated by an appropriate integration of the density functions of the random variables T_x and T_y

$$
{}_nq_{xy}^1 = \Pr[(x)\text{dies before } (y)\text{and within } n \text{ years}]
$$

$$
= \Pr\left[T_x < T_y \wedge T_x \leq n\right] = \int_{t=0}^{t=n} \int_{s=t}^{s=\infty} {}_tp_x \ \mu_{x+t} \ {}_sp_y \ \mu_{y+s} ds dt
$$

$$
{}_nq_{xy}^2 = \Pr[(x)\text{dies after } (y)\text{and within } n \text{ years}]
$$

$$
= \Pr\left[T_y < T_x \leq n\right] = \int_{t=0}^{t=n} \int_{s=0}^{s=t} {}_tp_x \ \mu_{x+t} \ {}_sp_y \ \mu_{y+s} ds dt
$$

Exercise: Derive that ${}_nq_{xy}^1 = \int_0^n {}_tp_{xy}\mu_{x+t} dt$

Solution: ${}_nq_{xy}^{1} = \Pr[(x)\text{dies before }(y)\text{and within }n\text{ years}] =$

$$\Pr\left[T_x < T_y \wedge T_x \leq n\right] = \int_{t=0}^{t=n} \int_{s=t}^{s=\infty} {}_tp_x \; \mu_{x+t} \; {}_sp_y \; \mu_{y+s} ds dt$$

$$= \int_{t=0}^{t=n} {}_tp_x \; \mu_{x+t} \underbrace{\left\{ \int_{s=t}^{s=\infty} {}_sp_y \; \mu_{y+s} ds \right\}}_{{}_tp_y} dt$$

$$= \int_{t=0}^{t=n} {}_tp_x \; \mu_{x+t} \; {}_tp_y \; dt = \int_0^n {}_tp_{xy} \mu_{x+t} dt$$

Exercise:

Justify that ${}_{n}q_{xy}^{2} = {}_{n}q_{x} - {}_{n}q_{xy}^{1}$, that is, “second death” probabilities can always be expressed in terms of “single death” and “first death” probabilities (this provides a method of evaluating “second death” probabilities).

Solution:

Straightforward, noting that (obviously, under this framework)

$${}_nq_x = {}_nq^1_{xy} + {}_nq^2_{xy}.$$

Similarly

$${}_nq^2_{xy} = {}_nq_y - {}_nq^1_{xy}$$

Exercise:

Justify that

$$ {}_{n}q_{xy}^{\;\;2} = {}_{n}q_{\overset{1}{x}y} - {}_{n}p_{x}\ {}_{n}q_{y}\ , $$

$$ \begin{aligned} {}_{n}q_{x\overset{1}{y}} &= \Pr[(y)\text{ dies before }(x)\text{ and within }n\text{ years}] \\ &= \Pr\left[T_y < T_x \wedge T_y \leq n\right]. \end{aligned} $$

Solution:

Also straightforward, noting that

$$\Pr[(x)\text{dies after } (y) \text{ and within } n \text{ years}]$$

$$= \Pr[(y) \text{ dies before } (x) \text{ and within } n \text{ years}]$$

$$-\Pr[(y) \text{ dies within } n \text{ years and } (x) \text{ survives } n \text{ years }]$$

$$_{n}q_{xy}^{\;2} \neq \; _{n}q_{xy}^{\;\;1}$$

5.8.8.2 EPV of contingent assurances

Contingent events depending on the future lifetime of two lives (x) and (y) can be written in terms of the random variables T_x and T_y. The random variables representing the present value of contingent assurances can be expressed as functions of these two random variables.

For example the present value of a sum assured of 1 paid immediately on the death of (x) provided that (y) is still alive can be expressed as

$$\bar{Z} = \begin{cases} v_i^{T_x} & if\ T_x \leq T_y \\ 0 & if\ T_x > T_y \end{cases}$$

where i is the valuation rate of interest.

Using similar methods to those used before, it follows that the mean of $\bar{Z}$ is

$$E[\bar{Z}] = \bar{A}_{\underset{xy}{1}} = \int_{t=0}^{t=\infty} v^t \; {}_tp_{xy}\, \mu_{x+t} dt,$$

as ${}_{\infty}q_{xy}^{1} = \int_{t=0}^{t=\infty} {}_tp_{xy}\, \mu_{x+t} dt$.

The variance of $\bar{Z}$ is

$$Var(\bar{Z}) = {}^2\bar{A}_{\underset{xy}{1}} - \left(\bar{A}_{\underset{xy}{1}}\right)^2$$

where ${}^2\bar{A}_{\underset{xy}{1}}$ is evaluated at a valuation rate of interest $i^2 + 2i$.

These functions are usually evaluated by using numerical methods to determine the values of the integrals. In some cases joint and single life values obtained from tables can be useful in conjunction with the following and similar relationships

$$\bar{A}_{xy} = \bar{A}_{\overset{1}{x}y} + \bar{A}_{x\overset{1}{y}}$$

$$\bar{A}_{x} = \bar{A}_{\overset{1}{x}y} + \bar{A}_{\overset{2}{x}y}$$

$$\bar{A}^{1}_{x:x} = \bar{A}_{x:\overset{1}{x}} = 0.5\bar{A}_{x:x}$$

$$\bar{A}^{2}_{x:x} = \bar{A}_{x:\overset{2}{x}} = 0.5\bar{A}_{\overline{x:x}}$$

(In these last results we assume the two lives are of the same age and subject to the same mortality model, although independent in respect of mortality)

If the benefit is payable at the end of the policy year in which the contingent event occurs then we can show that

$$A_{\overset{1}{xy}} = \sum_{k=0}^{\infty} v^{k+1} \, {}_{k}p_{xy} \, q_{x+k:y+k}^{1}$$

with analogous expressions for the variance to those derived for assurances payable immediately on the occurrence of the contingent event.

The approximate relationship

$$\bar{A}_{\overset{1}{xy}} \approx (1+i)^{\frac{1}{2}} A_{\overset{1}{xy}}$$

Is still valid.

5.8.8.3 EPV of reversionary whole life annuities

The simplest form of a reversionary annuity is one that begins on the death of (*x*), if (*y*) is then alive, and continues during the lifetime of (*y*). The life (*x*) is called the counter or failing life, and the life (*y*) is called the annuitant. The random variable $\bar{Z}$ representing the present value of this reversionary annuity if it is payable continuously can be written as a function of the random variables T_x and T_y , where

$$\bar{Z} = \begin{cases} \bar{a}_{\overline{T_y|}} - \bar{a}_{\overline{T_x|}} & if\ T_y > T_x \\ 0 & if\ T_y \le T_x \end{cases}$$

$$\bar{a}_{x|y} = E[\bar{Z}] = \bar{a}_y - \bar{a}_{xy} = \frac{\bar{A}_{xy} - \bar{A}_y}{\delta} = \int_{t=0}^{t=\infty} v^t \bar{a}_{y+t}\ {}_t p_{xy} \mu_{x+t} dt$$

If the annuity begins at the end of the year of death of (*x*) and is then paid annually in arrear during the lifetime of (*y*), the random variable, *Z* representing the present value of the payments can be written as a function of K_x and K_y , where

$$Z = \begin{cases} a_{\overline{K_y|}} - a_{\overline{K_x|}} & if\ K_y > K_x \\ 0 & if\ K_y \leq K_x \end{cases}$$

$$a_{x|y} = E[Z] = a_y - a_{xy} = \frac{A_{xy} - A_y}{d}$$

5.8.8.4 EPV of contingent assurances which depend upon term

Only term assurances are meaningful in this context. The expected present value of an assurance payable immediately on the death of (*x*) within *n* years provided (*y*) is then alive can be written

$$\bar{A}^{\;1}_{xy:\overline{n|}} = \int_{t=0}^{t=n} v^t \; {}_tp_{xy}\, \mu_{x+t} dt$$

with a similar expression involving summation operators if the sum assured is payable at the end of the year of death.

5.8.8.5 EPV of reversionary annuities which depend upon term

1. Annuity payable after a fixed term has elapsed:

A reversionary annuity in which the counter or failing status is a fixed term of *n* years **is exactly equivalent to a deferred life annuity**. The expected present value of an annuity which is paid continuously can be written

$$\bar{a}_{\overline{n|}|y} = \bar{a}_y - \bar{a}_{y:\overline{n|}}$$

(by analogy with $\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}$)

2. Annuity payable to *(y)* on the death of *(x)*, but ceasing at time *n*:

A reversionary annuity that ceases in any event after *n* years i.e. is payable to (*y*) after the death of (*x*) with no payment being made after *n* years.

$$EPV = \bar{a}_{y:\overline{n|}} - \bar{a}_{xy:\overline{n|}}$$

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3. Annuity payable to (*y*) on the death of (*x*), provided that (*x*) dies within *n* years, and then continues until the death of (*y*)

A reversionary annuity in which the payment commences on the death of (*x*) within *n* years and then continues until the death of (*y*).

$$EPV = \int_{t=0}^{t=n} v^t \ {}_tp_{xy}\, \mu_{x+t} \bar{a}_{y+t} dt$$

$$= \underbrace{\bar{a}_y - \bar{a}_{xy}}_{\bar{a}_{x|y}} - v^n \ {}_np_{xy} \underbrace{\left(\bar{a}_{y+n} - \bar{a}_{x+n:y+n}\right)}_{\bar{a}_{x+n|y+n}}$$

4. Annuity payable to (*y*) on the death of (*x*), for a maximum of *n* years

A reversionary annuity in which the payment will:

- Begin on the death of (*x*) and
- Cease on the death of (*y*) or *n* years after the death of (*x*) (whichever event occurs first)

$$EPV = \int_{t=0}^{t=\infty} v^t \ {}_tp_{xy}\, \mu_{x+t} \bar{a}_{y+t:\overline{n|}} dt$$

$$= \bar{a}_{x|y} - v^n \ {}_np_y \ \bar{a}_{x|y+n} = \bar{a}_y - \bar{a}_{xy} - v^n \ {}_np_y \ \bar{a}_{x|y+n}$$

$$= \bar{a}_y - \bar{a}_{xy} - v^n \ {}_np_y \ \left(\bar{a}_{y+n} - \bar{a}_{x:y+n}\right)$$

In this case, the *n* years of the term start counting on the death of (*x*). In Case 2., the *n* years of the term start counting at *t* = 0 .

5. Annuity payable to (y) on the death of (x) and guaranteed for n years (and then continues until the death of (y), in case (y) survives the guaranteed period)

$$EPV = \int_{t=0}^{t=\infty} v^t \ {}_tp_{xy}\, \mu_{x+t}\left(\bar{a}_{\bar{n}|} + v^n \ {}_np_{y+t}\bar{a}_{y+t+n}\right)dt$$

$$= \left(\bar{A}^1_{xy}\right)\bar{a}_{\bar{n}|} + v^n \ {}_np_y \ \bar{a}_{x|y+n}$$

An useful result when both lives are aged x and subject to the same mortality model:

$$\bar{A}^1_{x:x} = \bar{A}_{x:\overset{1}{x}} = 0.5\bar{A}_{x:x}$$

6. Annuity payable to (y) on the death of (x), and continuing for n years after (y)'s death

$$EPV = \bar{a}_{x|y} + \left(\bar{A}_{x:\overset{2}{y}}\right)\bar{a}_{\bar{n}|}$$

The first term is the EPV of the benefit payable after the death of (x) while (y) is still alive. The second term is the EPV of the annuity paid n years following the death of (y), provided that (y) dies after (x).

An useful result when both lives are aged x and subject to the same mortality model:

$$\overline{A}_{x:\overset{2}{x}} = \overline{A}^{2}_{x:x} = 0.5\bar{A}_{\overline{x:x}}$$

Similar expressions involving summation operators can be developed if the annuity payments are made at annual intervals from the date on which the counter status fails.

Example

A single premium special reversionary annuity policy is issued to a husband aged exactly 70 and a wife aged exactly 67. The annuity commences immediately on the death of the first of the lives to die and is payable subsequently while the second life is alive, for a maximum period of 20 years after the commencement date of the policy. The annuity, of annual amount 24000, is payable annually in advance. Calculate the single premium using the following basis:

Mortality: PMA92C20 for the male life and PFA92C20 for the female life. The lives are independent with respect to mortality.

Interest: 4% per annum.

Expenses: initial expenses of 500 incurred at the outset; annuity expenses of 1.5% per annum of the annuity payment, while the annuity is being paid.

Using the equivalence principle, the gross single premium is

$$P = 24\,000\left(\ddot{a}^{hw}_{\overline{70:67:\overline{20|}}} - \ddot{a}^{hw}_{70:67:\overline{20|}}\right) + 0.015 \times 24\,000\left(\ddot{a}^{hw}_{\overline{70:67:\overline{20|}}} - \ddot{a}^{hw}_{70:67:\overline{20|}}\right) + 500$$

$$\Leftrightarrow P$$

$$= 24\,000\left(\ddot{a}^{h}_{70:\overline{20|}} + \ddot{a}^{w}_{67:\overline{20|}} - 2\ddot{a}^{hw}_{70:67:\overline{20|}}\right) + 0.015 \times 24\,000\left(\ddot{a}^{h}_{70:\overline{20|}} + \ddot{a}^{w}_{67:\overline{20|}} - 2\ddot{a}^{hw}_{70:67:\overline{20|}}\right)$$

$$+ 500$$

$$\ddot{a}^{h}_{70:\overline{20|}} = \ddot{a}^{h}_{70} - v^{20}\frac{l^h_{90}}{l^h_{70}}\ddot{a}^{h}_{90} = 11.562 - 0.45639\frac{2675.203}{9218.134} \times 4.527 = 11.562 - 0.5996$$

$$= 10.9624$$

$$\ddot{a}^{w}_{67:\overline{20|}} = \ddot{a}^{w}_{67} - v^{20}\frac{l^w_{87}}{l^w_{67}}\ddot{a}^{w}_{87} = 14.111 - 0.45639\frac{5349.595}{9605.483} \times 6.582 = 14.111 - 1.673 = 12.438$$

$$\ddot{a}^{hw}_{70:67:\overline{20|}} = \ddot{a}^{hw}_{70:67} - v^{20}\frac{l^h_{90}}{l^h_{70}}\frac{l^w_{87}}{l^w_{67}}\ddot{a}^{h}_{90:87} = 10.233 - 0.45639 \times 0.16163 \times 3.528$$

$$= 10.233 - 0.2602 = 9.9728$$

$$P$$

$$= \underbrace{24\,000(10.9624 + 12.438 - 2 \times 9.9728)}_{82\,915.2} + \underbrace{0.015 \times 24\,000\left(\ddot{a}^{h}_{70:\overline{20|}} + \ddot{a}^{w}_{67:\overline{20|}} - 2\ddot{a}^{hw}_{70:67:\overline{20|}}\right)}_{1243.728}$$

$$+ 500 = 84\,658.928.$$

5.8.8.6 EPV of reversionary annuities payable m times a year

$$a_{x|y}^{(m)} = a_y^{(m)} - a_{xy}^{(m)} \approx a_y - a_{xy}$$

$$a_{y:\overline{n|}}^{(m)} - a_{xy:\overline{n|}}^{(m)} \approx a_{y:\overline{n|}} - a_{xy:\overline{n|}} + \frac{m-1}{2m}\left({}_np_{xy} v^n - {}_np_y\, v^n \right)$$

In every case, similar expressions can be developed for annuities payable in advance and, letting $m \to \infty$, continuous annuities.

5.9 Multiple decrement models (competing risks, or competing causes of decrement) revisited - Multiple decrement tables

A multiple decrement model is characterized by having a single starting state and several exit states with a possible transition from the starting state to any of the exit states, but no further transitions.

Assuming the transition intensities as functions of age x are known, it is possible to calculate ${}_tp_x^{00}$ and ${}_tp_x^{0i}$ using numerical or analytic integration.

A useful way of determining the probabilities of the various events is to use a "multiple decrement table".

Def. 1: A multiple decrement table is a computational tool for dealing with a population subject to more than one independent decrement. For example, a whole life assurance might be subject to surrender as well as death.

Notation (extension of the single decrement life table approach):

$(al)_x$ = population at age *x*

α, β, γ, … the labels for the types of independent decrements to which the population is subject.

$(ad)_x^k$ = number of lives removed due to decrement k, $k =$ α, β, γ, …

$(aq)_x^k = \frac{(ad)_x^k}{(al)_x}$ = dependent rate of decrement k, $k =$ α, β, γ, …

q_x^k = independent rate of decrement k, $k =$ α, β, γ, …

Remark 2: In general, as the decrements are assumed to operate independently (for instance, it is reasonable to believe that death and surrender are independent decrements), the number of lives removed due to decrement "*k*" will depend on the preceding population $(al)_x$ **as well as** the numbers removed by every other decrement other than *k*.

The value of $(aq)_x^k$, the dependent rate of decrement *k* ($k =$ α, β, γ…) depends on the effect of the other decrements operating on the population.

The value of q_x^k, the independent rate of decrement *k* ($k =$ α, β, γ…) results from assuming that the decrement *k* is operating in isolation.

The dependent probability $(aq)_x^k$ is the probability that a life aged x in a particular state will be removed from that state between ages x and $x+1$ by the decrement k, in the presence of all other decrements (risks) in the population.

The independent probability q_x^k is the probability that a life aged x in a particular state will be removed from that state between ages x and $x+1$ by the decrement k, if k is the only decrement acting on the population.

All dependent quantities are denoted in ( )s, with their corresponding independent values being quoted without the brackets.

Probabilities can be evaluated by numerical solution of the Kolmogorov equations, as seen before. Those Kolmogorov equations themselves require values for the transition intensities and are not of much use in practice.

Another method is going to be seen next, using multiple decrement tables.

The two methods are consistent.

Remark 3: The multiple decrement table is a numerical representation of the development of the population, such that

$$(al)_{x+1} = (al)_x$$

− number of lives removed between ages x and $x + 1$ due to decrement α

− number of lives removed between ages x and $x + 1$ due to decrement β

− number of lives removed between ages x and $x + 1$ due to decrement γ

− …

− number of lives removed between ages x and $x + 1$ due to decrement φ

Or

$$(al)_{x+1} = (al)_x - (ad)_x^{\alpha} - (ad)_x^{\beta} - (ad)_x^{\gamma} - \cdots - (ad)_x^{\varphi}$$

Although

$$ {}_t(ap)_x = {}_tp_x^{\alpha}\ {}_tp_x^{\beta} \times \cdots \times {}_tp_x^{\varphi} $$

$$ \Rightarrow \quad (ap)_x = p_x^{\alpha} \times p_x^{\beta} \times \cdots \times p_x^{\varphi}, $$

It is usual to calculate the probability $(ap)_x$ departing from the relationship

$$ (al)_{x+1} = (al)_x - (ad)_x^{\alpha} - (ad)_x^{\beta} - \cdots - (ad)_x^{\varphi} $$

$$ \Rightarrow \quad (ap)_x = 1 - (aq)_x^{\alpha} - (aq)_x^{\beta} - \cdots - (aq)_x^{\varphi}, $$

Dividing both sides of the equation by $(al)_x$.

A multiple decrement model is really just a special case of a multiple state model.

For example, the probability that an individual who is in State 0 at age x is in State j at age $x+1$ is denoted by $(aq)_x^j$ or p_x^{0j}.

Another example: both $(ap)_x$ and p_x^{00} denote the probability that a life in State 0 at age x is in State 0 at age $x+1$.

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CASE 1:

Assuming that:

- **There are *m* decrements which are** (jointly) **uniformly distributed** over the year of age $(x, x+1)$ in the multiple decrement table.
- The *m* decrements are independent, that is, the rate of decrement from cause *j* is independent from the rate of decrement from cause *i*. (This is the linking assumption between single and multiple decrement tables).

The formula for obtaining independent rates from dependent rates

is $$q_x^k = \frac{(aq)_x^k}{1-\frac{1}{2}(aq)_x^{-k}} \left(= \frac{(ad)_x^k}{(al)_x-\frac{1}{2}(ad)_x^{-k}} \right),$$

where $(aq)_x^{-k}$ is the probability that a life aged *x* exact leaves the population over the next year of age due to all causes except cause k; Thus, if the causes $-k$ are eliminated, each one of these lives will be exposed to decrement from cause *k* for an average of ½ year in (*x*, *x*+1).

For two decrements, α and β:

$$q_x^{\alpha} = \frac{(aq)_x^{\alpha}}{1-\frac{1}{2}(aq)_x^{\beta}} \text{ and } q_x^{\beta} = \frac{(aq)_x^{\beta}}{1-\frac{1}{2}(aq)_x^{\alpha}}.$$

[From here we derive that $(aq)_x^{\alpha} = \frac{q_x^{\alpha}\left(1-\frac{1}{2}q_x^{\beta}\right)}{1-\frac{1}{4}q_x^{\alpha}q_x^{\beta}}$ and $(aq)_x^{\beta} = \frac{q_x^{\beta}\left(1-\frac{1}{2}q_x^{\alpha}\right)}{1-\frac{1}{4}q_x^{\alpha}q_x^{\beta}}$]

For three decrements, α, β and γ:

$$q_x^{\alpha} = \frac{(aq)_x^{\alpha}}{1-\frac{1}{2}\left[(aq)_x^{\beta}+(aq)_x^{\gamma}\right]} \text{ , } q_x^{\beta} = \frac{(aq)_x^{\beta}}{1-\frac{1}{2}\left[(aq)_x^{\alpha}+(aq)_x^{\gamma}\right]} \text{ and}$$

$$q_x^{\gamma} = \frac{(aq)_x^{\gamma}}{1-\frac{1}{2}\left[(aq)_x^{\alpha}+(aq)_x^{\beta}\right]}.$$

These formulae enable the underlying single decrement tables to be obtained from an estimated multiple decrement table.

Example

You are given the following extract from a double decrement table:

x	$(al)_x$	$(ad)_x^d$	$(ad)_x^w$
20	100,000	175	24,975
21	74,850	–	–

where d denotes death and w denotes withdrawal. Assuming that deaths and withdrawals are uniformly distributed over the year of age (20,21) in the double decrement table, calculate q_{20}^d and q_{20}^w.

Solution

Since deaths and withdrawals are uniformly distributed over the year of age (20,21) in the *multiple* decrement table:

$$q_{20}^{d} = \frac{(ad)_{20}^{d}}{(al)_{20} - \frac{1}{2}(ad)_{20}^{w}} = \frac{175}{100{,}000 - \frac{1}{2}\times 24{,}975} = 0.00200$$

$$q_{20}^{w} = \frac{(ad)_{20}^{w}}{(al)_{20} - \frac{1}{2}(ad)_{20}^{d}} = \frac{24{,}975}{100{,}000 - \frac{1}{2}\times 175} = 0.24997$$

Example

Suppose that the independent rate of mortality at age 20 in the example above falls from 0.00200 to 0.00100 but the independent rate of withdrawal is unchanged. Reconstruct the double decrement table to take account of the improvement in mortality.

Solution

We have:

$$0.00100 = \frac{(aq)_{20}^{d}}{1-\frac{1}{2}(aq)_{20}^{w}} \Rightarrow (aq)_{20}^{d} = 0.00100\left[1-\frac{1}{2}(aq)_{20}^{w}\right]$$

$$0.24997 = \frac{(aq)_{20}^{w}}{1-\frac{1}{2}(aq)_{20}^{d}} \Rightarrow (aq)_{20}^{w} = 0.24997\left[1-\frac{1}{2}(aq)_{20}^{d}\right]$$

Substituting the expression for $(aq)^w_{20}$ into the formula for $(aq)^d_{20}$ gives:

$$(aq)^d_{20} = 0.00100\left[1-\tfrac{1}{2}\times 0.24997\left(1-\tfrac{1}{2}(aq)^d_{20}\right)\right]$$

So:

$$(aq)^d_{20}\left[1-0.00100\times\tfrac{1}{2}\times 0.24997\times\tfrac{1}{2}\right] = 0.00100\left[1-\tfrac{1}{2}\times 0.24997\right]$$

$$\Rightarrow (aq)^d_{20} = 0.00088$$

and:

$$(aq)^w_{20} = 0.24997\left[1-\tfrac{1}{2}(aq)^d_{20}\right] = 0.24986$$

The new double decrement table is then:

x	$(al)_x$	$(ad)_x^d$	$(ad)_x^w$
20	100,000	88	24,986
21	74,926	–	–

Comparing this with the original table, we see that the number of deaths has fallen as we would expect. Notice also that the number of withdrawals has increased from 24,975 to 24,986. The fall in the mortality rate means that more people are now available to withdraw.

CASE 2 (page 35 orange book):

Assuming that:

- **There are *m* decrements $\boldsymbol{\alpha, \beta, \gamma, \dots, \tau, \varphi}$, and each one is uniformly distributed** over each year of age in the single decrement table.
- The *m* decrements are independent.

Then, for decrement α (and, an a similar way, for any other decrement)

$$(aq)_x^\alpha = \frac{(ad)_x^\alpha}{(al)_x} = \int_0^1 \frac{(al)_{x+t}}{(al)_x}(a\mu)_{x+t}^\alpha dt = \int_0^1 {}_t(ap)_x \; \mu_{x+t}^\alpha dt.$$

Note that $(a\mu)_x^k = \mu_x^k$ for all *k* and all *x* – linking assumption between single and multi decrement tables.

$$\int_0^1 {}_t(ap)_x \; \mu_{x+t}^{\alpha} dt = \int_0^1 \frac{{}_t(ap)_x}{{}_tp_x^{\alpha}} \; {}_tp_x^{\alpha} \mu_{x+t}^{\alpha} dt$$

and since ${}_t(ap)_x = {}_tp_x^{\alpha} \; {}_tp_x^{\beta} \times \cdots \times {}_tp_x^{\varphi}$, we can write

$$(aq)_x^{\alpha} = \int_0^1 {}_tp_x^{\beta} \times \cdots \times {}_tp_x^{\varphi} \; {}_tp_x^{\alpha} \mu_{x+t}^{\alpha} dt.$$

Under the UDD assumption, ${}_sp_x\mu_{x+s} = q_x, 0 \leq s \leq 1$, and then

$$(aq)_x^{\alpha} = \int_0^1 \left(1 - tq_x^{\beta}\right) \times \cdots \times \left(1 - tq_x^{\varphi}\right) q_x^{\alpha} \, dt$$

Further simplifications give the following result:

$$(aq)_x^{\alpha} = q_x^{\alpha} \Bigg(1 - \frac{1}{2}\left(q_x^{\beta} + q_x^{\gamma} + \cdots + q_x^{\tau} + q_x^{\varphi}\right)$$
$$+ \frac{1}{3}\left(q_x^{\beta} q_x^{\gamma} + q_x^{\beta} q_x^{\delta} + \cdots + q_x^{\tau} q_x^{\varphi}\right) - \frac{1}{4}\left(q_x^{\beta} q_x^{\gamma} q_x^{\delta} + \cdots\right) + \cdots \Bigg)$$

To illustrate, if the population is subject to two decrements α, β, then the equations to use are

$$(aq)_x^\alpha = q_x^\alpha\left(1 - \tfrac{1}{2} q_x^\beta\right) \qquad \text{and} \qquad (aq)_x^\beta = q_x^\beta\left(1 - \tfrac{1}{2} q_x^\alpha\right)$$

For the three decrements α, β, γ case:

$$(aq)_x^\alpha = q_x^\alpha\left(1 - \frac{1}{2}\left(q_x^\beta + q_x^\gamma\right) + \frac{1}{3} q_x^\beta q_x^\gamma\right)$$

$$(aq)_x^\beta = q_x^\beta\left(1 - \frac{1}{2}\left(q_x^\alpha + q_x^\gamma\right) + \frac{1}{3} q_x^\alpha q_x^\gamma\right)$$

$$(aq)_x^\gamma = q_x^\gamma\left(1 - \frac{1}{2}\left(q_x^\alpha + q_x^\beta\right) + \frac{1}{3} q_x^\alpha q_x^\beta\right)$$

If we know the independent decrement rates, we can calculate the dependent decrement rates and construct the multiple decrement table.

Remember: To apply the correct formulae, we must then pay attention to the case that is under consideration:

If decrements are uniformly distributed in the multiple decrement table (Case 1, two decrements), use formulae:

$$q_x^{\alpha} = \frac{(aq)_x^{\alpha}}{1-\frac{1}{2}(aq)_x^{\beta}} \text{ and } q_x^{\beta} = \frac{(aq)_x^{\beta}}{1-\frac{1}{2}(aq)_x^{\alpha}}$$

If each one of the decrements is uniformly distributed in the single decrement table (Case 2, two decrements) use formulae:

$$(aq)_x^{\alpha} = q_x^{\alpha}\left(1-\frac{1}{2}q_x^{\beta}\right) \text{ and } (aq)_x^{\beta} = q_x^{\beta}\left(1-\frac{1}{2}q_x^{\alpha}\right)$$

(If the dependent rates are given, to calculate the independent rates **with more than two decrements** it is necessary to solve a set of simultaneous equations by means of an iterative procedure.)

Q11 April 2011

A life insurance company issues a 4-year unit-linked endowment policy to a life aged 61 exact under which level premiums of £2,500 are payable yearly in advance

Independent rate of mortality	AM92 Select
Independent rate of surrender	6% per annum

(i) Construct a multiple decrement table for this policy assuming that there is a uniform distribution of both decrements over each year of age in the single decrement table. [3]

x	q_x^d	q_x^s
61	0.006433	0.06
62	0.009696	0.06
63	0.011344	0.06
64	0.012716	0.06

x	$(aq)_x^d$	$(aq)_x^s$
61	0.006240	0.05981
62	0.009405	0.05971
63	0.011004	0.05966
64	0.012335	0.05962

Remark (based on CT 5 exams):

Case 1: decrements are uniformly distributed in the multiple decrement table;

Alternative description: dependent decrements are uniformly distributed over the year of age (because only the dependent probabilities can be directly calculated from the multiple decrement table - and we use the formulae above to find the independent ones, see solutions paper CT5 exam April 2010 – Q9). We already know that we must use formulae (two decrements)

$$q_x^{\alpha} = \frac{(aq)_x^{\alpha}}{1-\frac{1}{2}(aq)_x^{\beta}} \text{ and } q_x^{\beta} = \frac{(aq)_x^{\beta}}{1-\frac{1}{2}(aq)_x^{\alpha}}$$

Remark (based on CT 5 exams):

Case 2: each one of the decrements is uniformly distributed in the single decrement table;

Alternative description): independent decrements are uniformly distributed over the year of age (because only the independent probabilities can be directly calculated from the single decrement tables - and we use the formulae above to find the dependent ones, *idem*). We already know that we must use formulae (two decrements)

$$(aq)_x^\alpha = q_x^\alpha\left(1 - \frac{1}{2}q_x^\beta\right) \text{ and } (aq)_x^\beta = q_x^\beta\left(1 - \frac{1}{2}q_x^\alpha\right)$$

In some (twisted) exercises the independent rates are given in **Case 1** or the dependent rates are the ones given in **Case 2**, so that to calculate the dependent/independent rates it is necessary to solve a set of simultaneous equations. In Case 2 with more than two decrements it is necessary to use an iterative procedure to solve the system. Formulae on page 35 of the orange book refer to Case 2 only (do not use them in Case 1).

CASE 3 (usual in CT5 exams): If we are given q_x^α and q_x^β and one of the two decrements (for instance, β) is assumed to occur only at the end of the year, then

$$(aq)_x^\alpha = q_x^\alpha \text{ and}$$

$$(aq)_x^\beta = q_x^\beta(1 - (aq)_x^\alpha)$$

Example: if there are two decrements, death and withdraw, and withdraws are allowed only at the end of each policy year, the probability of death in each year is not influenced by withdraws, because there aren't any along the year.

Remark to CASE 3: we have seen that if we are given q_x^α and q_x^β and one of the two decrements (for instance, β) is assumed to occur only at the end of the year, then

$$(aq)_x^\alpha = q_x^\alpha \text{ and}$$

$$(aq)_x^\beta = q_x^\beta(1-(aq)_x^\alpha) = q_x^\beta(1-q_x^\alpha).$$

This is different from the case where payment of the surrender amount is made at the end of the year, independently of when the withdraw occurred: in this case withdraws actually occur along the year, influencing the probability of death and thus making $(aq)_x^d \neq q_x^d$. Policyholders who die after surrendering are not accounted for. Under such conditions we are really in Case 2.

CASE 4: the **forces of decrement** are given.

IFoA Exam 7 October 2016 – Q13 (i)

A life insurance company issues a 3-year endowment assurance policy to an ummarried life.

The company assumes that:

- each force of decrement is independent and constant over each year of age.
- surrenders only occur in policy years 1 and 2.

(i) Determine for each policy year the dependent rates of mortality, marriage and surrender. [4]

Independent force of marriage	15%
Independent force of surrender	7.5% in years 1 and 2 only
Independent force of mortality	1%

The dependent rates of decrement are calculated for policy years 1 and 2 using:

$$(aq)_x^j = \frac{\mu^j}{\mu^d + \mu^m + \mu^s}\left[1 - e^{-(\mu^d + \mu^m + \mu^s)}\right]$$

where d denotes mortality, m marriage and s surrenders

$\Rightarrow$

$$(aq)_x^d = \frac{\mu^d}{\mu^d + \mu^m + \mu^s}\left[1 - e^{-(\mu^d + \mu^m + \mu^s)}\right] = \frac{0.01}{0.235}\left[1 - e^{-(0.235)}\right] = 0.008912$$

$$(aq)_x^m = \frac{\mu^m}{\mu^d + \mu^m + \mu^s}\left[1 - e^{-(\mu^d + \mu^m + \mu^s)}\right] = \frac{0.15}{0.235}\left[1 - e^{-(0.235)}\right] = 0.133678$$

$$(aq)_x^s = \frac{\mu^s}{\mu^d + \mu^m + \mu^s}\left[1 - e^{-(\mu^d + \mu^m + \mu^s)}\right] = \frac{0.075}{0.235}\left[1 - e^{-(0.235)}\right] = 0.066839$$

The dependent rates of decrement are calculated for policy year 3 using:

$$(aq)_x^j = \frac{\mu^j}{\mu^d + \mu^m}\left[1 - e^{-(\mu^d + \mu^m)}\right]$$

where d denotes mortality and m marriage

$\Rightarrow$

$$(aq)_x^d = \frac{\mu^d}{\mu^d + \mu^m}\left[1 - e^{-(\mu^d + \mu^m)}\right] = \frac{0.01}{0.16}\left[1 - e^{-(0.16)}\right] = 0.009241$$

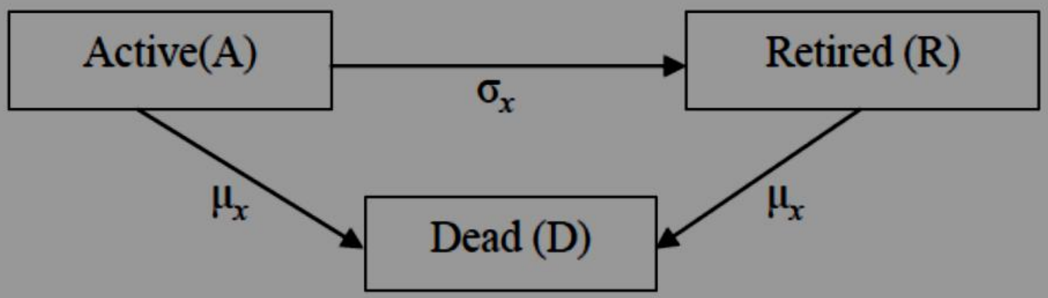
$$(aq)_x^m = \frac{\mu^m}{\mu^d + \mu^m}\left[1 - e^{-(\mu^d + \mu^m)}\right] = \frac{0.15}{0.16}\left[1 - e^{-(0.16)}\right] = 0.138615$$

Multiple decrement table:

t	$(aq)_{x+t}^{d}$	$(aq)_{x+t}^{m}$	$(aq)_{x+t}^{s}$
1	0.008912	0.133678	0.066839
2	0.008912	0.133678	0.066839
3	0.009241	0.138615	0.00

Exam April 2010 – Q9

9 A life insurance company models the experience of its pension scheme contracts using the following three-state model:



(i) Derive the dependent probability of a life currently Active and aged x retiring in the year of age x to $(x + 1)$ in terms of the transition intensities. [2]

(ii) Derive a formula for the independent probability of a life currently Active and aged x retiring in the year of age x to $(x + 1)$ using the dependent probabilities. [4]

[Total 6]

Exam April 2014 – Q8

8 A double decrement table is to be constructed from two single decrement tables. The modes of decrement are α and β. The basis for each of the single decrement tables is shown below:

Basis:

In the table for single decrement α: $l^{\alpha}_{x+t} = l^{\alpha}_{x} - t^3 d^{\alpha}_{x}$ for $0 \le t \le 1$

In the table for single decrement β: $l^{\beta}_{x+t} = l^{\beta}_{x} - t^5 d^{\beta}_{x}$ for $0 \le t \le 1$

The l function represents the number of lives and the d function the number of decrements in the appropriate table.

(i) Show that

$$ {}_tp^{\beta}_{x}\mu^{\beta}_{x+t} = 5t^4 q^{\beta}_{x} \text{ for } 0 \le t \le 1 $$ [3]

(ii) Hence or otherwise show that

$$ (aq)^{\beta}_{x} = q^{\beta}_{x}\left(1 - \frac{5}{8} q^{\alpha}_{x}\right) $$ [5]

[Total 8]

SURVIVAL MODELS AND LIFE CONTINGENCIES

MASTER IN ACTUARIAL SCIENCE

LECTURE 26

6. DISCOUNTED EMERGING COST TECHNIQUES
Dickson *et al.* – Chaps. 11 and 12 pp. 353-400

Institute and Faculty of Actuaries,

Subject CM1
Actuarial Mathematics Core Principles
Core Reading

UNIT 6 – PRICING AND RESERVING (**3. Profit testing)**

6.1 Evaluating expected cash flows

The standard approach to evaluate expected cash flows is to divide the total duration of a contract into a series of non-overlapping time periods.

- The length of each time period is chosen so that it is reasonable to make simple assumptions about the cash flows within each period e.g. funds earn a constant rate of interest during the period, a particular cash flow accrues uniformly during the period. These assumptions allow the expected cash flows during the period to be evaluated.
- The arithmetic of these calculations is usually most straightforward when the expected cash flows per contract in force at the start of the time period are calculated.

- The expected cash flows are used to construct a projected revenue account (per contract in force at the start of the period) for each time period. For some contracts e.g. life assurance, there is only one way in which a contract remains in force and so only one projected revenue account is needed. For other contracts e.g. disability insurance, there is more than one way in which the contract remains in force e.g. policyholder alive and not receiving disability benefits, policyholder alive and receiving disability benefits.

- Each in force status generates different cash flows in the subsequent time period and so separate projected revenue accounts are needed for each in-force status. **The balancing item in the projected revenue account is the profit emerging at the end of the time period.**

6.2 Deterministic profit testing for traditional life insurance

6.2.1 Required information

In order to calculate the expected cash flows the following information is needed:

- premiums paid and their times of payment
- expected expenses (from basis) and their times of payment
- contingent benefits payable under the contract e.g. death benefit, annuity payment, survival benefit for endowment…
- other benefits payable under the contract e.g. surrender values
- other expected cash payments e.g. taxes
- other expected cash receipts
- the reserves required for a contract, usually at the beginning and end of the time period, calculated using the valuation basis
- the different probabilities of the various events leading to the payment of particular cash amounts.

Remark 4: In modelling cash flows, we use reserves rather than policy values. The reserve is the actual amount of money held by the insurer to meet future liabilities and may be equal to the policy value or of a different amount.

Since the reserves are amounts that the insurer needs to assign from its assets to support the policy, it is necessary to include in profit testing the cost of assigning these amounts.

Usually, though, for traditional insurance, the policy value calculation will be used to set reserves, perhaps using a conservative basis

Remark 5: Any balance on the expected revenue account during the time period will be invested, and an assumption about the rate of return on these funds is needed. This allows the expected investment income during the period to be calculated and credited at the end of the period.

6.2.2 Illustrations

6.2.2.1 Whole life assurance

The contract is issued to a select life aged x and has a sum assured of S secured by level annual premiums of P.

The premium basis assumes initial expenses of I and renewal expenses of e.

The valuation basis requires reserves of $S\ {}_tV$ for an in-force policy with sum assured S at policy duration t.

The basis assumes that invested funds earn an effective rate i.

The surrender value basis determines that surrender of amount $(SV)_t$ will be paid to policies surrendered at policy duration t.

The probabilities of events are determined from a multiple decrement table with decrements of death, d, and surrender, w, having dependent rates at age x of $(aq)_x^d$ and $(aq)_x^w$.

Income	
Premiums (from data)	P
Interest on Reserves	$iS\ {}_tV$
Interest on Balances	$(P-e)i$
Expenditure	
Expenses (from data)	e
Expected Surrender Value	$(aq)^w_{[x]+t}(SV)_{t+1}$
Expected Death Claims	$(aq)^d_{[x]+t}S$
Transfer to Reserves	$(ap)_{[x]+t} \times \mathrm{S} \times {}_{t+1}V - \mathrm{S}\ {}_tV$
Profit	Balancing item

6.2.2.2 Disability insurance with waiver of premium

The contract is issued to a life aged *x* and is secured by level annual premiums of *P* which are waived during periods of disability.
The premium basis assumes initial expenses of *I* and renewal expenses of *e*.
Benefits of *S* p.a. are paid weekly during periods of disability.
The valuation basis requires reserves of ${}_tV^H$ and ${}_tV^S$ for policies not receiving and receiving benefits respectively at policy duration *t*.
The basis assumes that invested funds earn an effective rate *i*.
The policy does not acquire a surrender value at any time.
The probabilities of events for disabled lives are determined from a multiple decrement table with decrements of recovery from sickness, *r*, and death, *d*, having dependent rates at age *x* of $(sq)_x^r$ and $(sq)_x^d$.
The dependent rates of dying and falling sick for healthy lives at age *x* are $(hq)_x^s$ and $(hq)_x^d$ respectively.

Then for a life who is sick at the beginning of the (t +1)th policy year (time t) the projected revenue account is

Income	
Interest on reserves	$i\,{}_tV^S$
Interest on expenses	$-ie$
Expenditure	
Expenses (from data)	e
Expected Sickness Payment (ear-end)	$S\left(1-\frac{1}{2}(sq)^r_{x+t}-\frac{1}{2}(sq)^d_{x+t}\right)(1+i)^{\frac{1}{2}}$
Transfer to Reserves	$(sq)^r_{x+t}\,{}_{t+1}V^H+\left(1-(sq)^r_{x+t}-(sq)^d_{x+t}\right){}_{t+1}V^S-\ {}_tV^S$
Profit	Balancing item

For a life who is healthy at the beginning of the $(t+1)$th policy year (time t) the projected revenue account is

Income	
Premiums (from data)	P
Interest on reserves	$i\ {}_tV^H$
Interest on balances	$(P-e)i$
Expenditure	
Expenses (from data)	e
Expected Sickness Payment (revalued to year-end)	$\frac{S}{2}(hq)^s_{x+t}(1+i)^{\frac{1}{2}}$
Transfer to Reserves	$(hq)^s_{x+t\ t+1}V^s+\left(1-(hq)^s_{x+t}-(hq)^d_{x+t}\right)_{t+1}V^H - {}_tV^H$
Profit	Balancing item

6.DISCOUNTED EMERGING COST TECHNIQUES (Dickson *et al.* – Chaps. 11 and 12 pp. 353-400; Institute and Faculty of Actuaries, **Subject CM1 Actuarial Mathematics Core Principles** - *Core Reading*, UNIT 6 – PRICING AND RESERVING)

6.2 Deterministic profit testing for traditional life insurance
6.2.3 Profit tests for annual premium contracts
6.2.3.1 Profit Vector and Profit Signature

The purpose of a profit test is to identify the profit which the insurer can claim from the contract at the end of each time period.

The first step in the profit testing of a contract is the construction of the projected revenue accounts for each policy year.

All cash flows related to the policy are the components of the projected revenue account.
The calculation of the direct cash flows will also require:
data items about the contract e.g. initial and renewal expenses;
assumptions to form a basis e.g. mortality of policyholders, rate of return earned.

Def. 6:
The vector of balancing items in the projected revenue accounts for each policy year is called the **profit vector**,

$(PRO)_t = (PRO_1, PRO_2, \ldots, PRO_n)'$.

The profit vector gives the expected profit at the end of each policy year per policy in force at the beginning of that policy year. For some contracts the expected profit will depend upon the policyholder's status at the beginning of the policy year e.g. receiving or not receiving sickness benefit.

(Other notation: $\mathrm{Pr} = (\mathrm{Pr}_1, \mathrm{Pr}_2, \ldots, \mathrm{Pr}_n)'$).

Remark 7: In many cases,

$$Pr_t = \left({}_{t-1}V + P - e_t\right)(1+i) - Sq_{x+t-1} - {}_tV \times p_{x+t-1},$$

equivalent to

$$Pr_t = (P - e_t)(1+i) + \Delta\ {}_tV - Sq_{x+t-1},$$

$\Delta\ {}_tV = (1+i)_{t-1}V - {}_tV \times p_{x+t-1}$ being the **change in reserve** (or **cost of increase in reserve**) at time t.

This alternative expression reflects the difference between the reserves and the other cash flows. The incoming and outgoing reserves each year are not real income and outgo in the same way as premiums, claims and expenses, but accounting transfers.

Def. 8:

The vector of expected profits per policy issued is called the **profit signature**, $(PS)_t = (PS_1, PS_2, \dots, PS_n)'$. This is obtained by using transition probabilities from policy duration 0 to policy duration $t-1$. (Other notation: $\Pi = (\Pi_1, \Pi_2, \dots, \Pi_n)'$).

Example:

Life assurance $(PS)_t = {}_{t-1}p_x \ (PRO)_t$

Disability assurance $(PS)_t = {}_{t-1}p_x^{HH}(PRO)_t^H + {}_{t-1}p_x^{HS}(PRO)_t^S$,
where H = healthy (not receiving benefit) and S = sick (receiving benefit).

The profit vector is the vector of expected end-year profits for policies which are still in force at the start of each year.

The profit signature is the vector of expected end-year profits allowing for survivorship from the start of the contract.

Remark 9:

- The vector representing the profit signature $(PS)_t$ can be displayed graphically to illustrate the way in which profits are expected to emerge over the lifetime of the policy.

- It is difficult to compare this information for different policies when there is a need to evaluate alternative designs for a product (policy) or to decide which of several different possible policies is the most profitable.

- Decisions like this are usually made easier by summarising each profit signature as a single figure.

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6.2.4. Summary measures of profit

Summary measures usually involve determining the present values of cash flows. This requires an assumption about the discount rate, called the risk discount rate,

$$i_d = \text{cost of capital}$$

$$+$$

premium to reflect the risks and uncertainties surrounding the cash flows to and from the policy

The cost of capital is the rate at which funds can be borrowed, if this is necessary, or the rate which funds would otherwise earn if they are to be diverted from alternative investment opportunities.

Def. 10: Net present value (NPV)

This is the present value of the profit signature determined using the risk discount rate.

Def. 11: Profit margin

This is the expected NPV of the profit signature expressed as a percentage of the expected net present value of the premium income.

If the premium paid at the beginning of the t-th policy year is P_t, then

$$Profit\ margin = \frac{\sum_{t=1}^{n} \frac{(PS)_t}{(1+i_d)^t}}{\sum_{t=1}^{n} \frac{{}_{t-1}p_x P_t}{(1+i_d)^{(t-1)}}}$$

Remark 12:

Other usual profit measures: The IRR and the DPP

The **internal rate of return (IRR)** is the interest rate j such that the present value of the expected cash flows is zero. Given a profit signature $(PS)_t\,, t\ =\ 1, 2, 3, \dots, n,$ for an n-year contract, the internal rate of return is j where

$$\sum_{t=0}^{n} PS_t\, v_j^t = 0$$

The **discounted payback period (DPP)**, also known as the break-even period is calculated using the risk discount rate i_d and is the smallest value of m such that

$$\sum_{t=0}^{m} PS_t\, v_{i_d}^t \geq 0$$

The DPP represents the time until the insurer starts to make a profit on the contract.

6.2.5 Pricing and reserving using profit testing

If the premiums for a contract together with all the other data items about the contract are known, then given a basis on which the projected revenue accounts can be calculated, the expected profitability of the contract can be evaluated.

Of course, the actual profitability is an unknown quantity until each respective contract terminates and the actual experience becomes known.

Example: Q13 CT5 exam - October 2014

SURVIVAL MODELS AND LIFE CONTINGENCIES

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6.3 Deterministic profit testing for equity-linked insurance

6.3.1 Equity linked insurance

In some modern insurance contracts the main purpose is investment. These contracts include some life contingent guarantees, predominantly as a way of distinguishing them from pure investment products.

Modern life insurance products are usually more flexible and often involve an investment component. The table below summarizes the features of several modern life insurance products.

Product	**Features**
Universal life insurance	– Combines investment and life insurance – Premiums are flexible, as long as the accumulated value of the premiums is enough to cover the cost of insurance
Unitized with-profit insurance insurance	– Similar to traditional participating insurance – Premiums are used to purchase shares of an investment fund. – The income from the investment fund increases the sum insured.
Equity-linked insurance	– The benefit is linked to the performance of an investment fund. – Examples: equity-indexed annuities (EIA), unit-linked policies, segregated fund policies, variable annuity contracts – Usually, investment guarantees are provided.

- SOURCE: (2015) ACTEX MLC STUDY MANUAL

Such contracts are called **unit-linked insurance** in the UK and parts of Europe, **variable annuities** in the USA (though there is often no annuity component) and **segregated funds** in Canada. All fall under the generic title of equity-linked insurance.

The basic premise of equity-linked insurance is that a policyholder pays a single or regular premium which, after deducting expenses, is invested on the policyholder's behalf. The accumulating premiums form the **policyholder's fund**. Regular **management charges** are deducted from the fund by the insurer and paid into the **insurer's fund** to cover expenses and insurance charges.

On survival to the end of the term the benefit may be just the policyholder's fund and no more, or there may be a **guaranteed minimum maturity benefit** (GMMB).

On death during the term of the policy, the policyholder’s estate would receive the policyholder’s fund, possibly with an extra amount – for example, a death benefit of 110% of the policyholder’s fund means an additional payment of 10% of the policyholder’s fund at the time of death. There may also be a **guaranteed minimum death benefit** (GMDB).

That is:

- **Unit-linked assurances** (typically whole life or endowment) have benefits which are directly linked to the value of the underlying investments.

- Each policyholder receives the value of the units allocated to the policy. There is no pooling of investments or allocation of the pooled surplus.

- As each premium is paid, a specified proportion (the “allocation percentage”) is invested in an investment fund chosen by the policyholder. The investment fund is divided into units which are priced continuously.

The value at the date of death or survival of the cumulative number of units purchased is the sum assured under the policy.

Sometimes a minimum guaranteed sum assured is specified in the contract to ensure that the policyholder avoids any difficulties arising from a particularly poor investment performance.

In order to price and value unit-linked contracts details of allocation percentages (usually specified in the policy) and an assumption about the future growth in the price of the units purchased are needed.

Important terminology:

Unit account: the total value of the units in respect of the policy at any time.

Allocation percentage (example): if 90% of the premium is allocated to units, then 90% of the premium goes to the policyholder’s fund and the rest goes to the insurer’s fund.

Bid-offer price (spread) (example): If the previous contract is sold with a bid-offer spread of, say, 5%, then only 95% of the allocated premium is actually invested in the policyholder’s fund; the remainder goes to the insurer’s fund.

In short: To the policyholder’s fund goes 95% of 90% that is 85.5% of the full premium; the remaining 14.5% goes to the insurer’s fund.

Charges: the company will deduct money from the unit account on a periodic basis, in respect of expenses and the cost of providing cover in respect of any contingency.

Equity-linked insurance policies are also usually analysed using emerging surplus techniques applied last day (the process of projecting the income and outgo emerging from a policy, and discounting the results).

The cash flows can be separated into those that are in the policyholder’s fund and those that are income or outgo for the insurer.

It is the insurer’s cash flows that are important in pricing and reserving, but since the insurer’s income and outgo depend on how much is in the policyholder’s fund, <u>we must first project the cash flows for the policyholder’s fund and then use these to project the cash flows for the insurer’s fund</u>.

The projected cash flows for the insurer’s fund can then be used to calculate the profitability of the contract using the profit vector, profit signature, and perhaps the NPV, IRR, profit margin and discounted payback period, in the same way as before.

That is: the most important thing to bear in mind with unit-linked contracts is that it is necessary now keep track of two worlds: the “unit world” and the “cash world”.

Unit world (the unit fund): The policyholder pays premiums to acquire units, and the eventual benefit is normally denominated in these units, so it is necessary to keep track of the number of units bought, how they are growing, and what charges are being deducted from them.

Cash world (the non-unit fund): the policyholder pays the insurer in money. So it is necessary to keep track of the cash not used to buy units. If the policyholder dies there might be a cash denominated sum insured, so it is necessary to keep track of the cash outgo on claims (any sum insured payable on death in excess of the value of the units, or any guaranteed maturity value in excess of the value of the units, are non-unit benefits, come from the non-unit fund). The company’s expenses (underwriting and maintaining expenses and commissions) are another important cash outgo.

To retain:

- The unit fund is worth only the bid value of the allocated premium – everything else in the premium goes to the non-unit fund.
- The charges and what they represent are different: the charge for cost of cover could be different from the actual cost of cover and the charge for fund management expenses could be different from the actual fund management expenses.
- The profit or loss to the insurer in each year will be the balance in the non-unit fund between all sources of income (charges, unallocated premium, bid/offer spread) and all sources of outgo (expenses, non-unit benefits).
- The unit fund is what the policyholder sees (unit growth and all charges are communicated). The non-unit fund is what goes within the company, and the policyholder does not see anything at this level.

So, the main features of a unit-linked policy are:

Allocated premiums are invested in a fund (or funds) chosen by the policyholder which purchases a number of units within that fund (funds).

Each investment fund is divided into units, which are priced regularly (usually daily).

Policyholder receives the value of the units allocated to his/her own policy.

Benefits are directly linked to the value of the underlying investments.

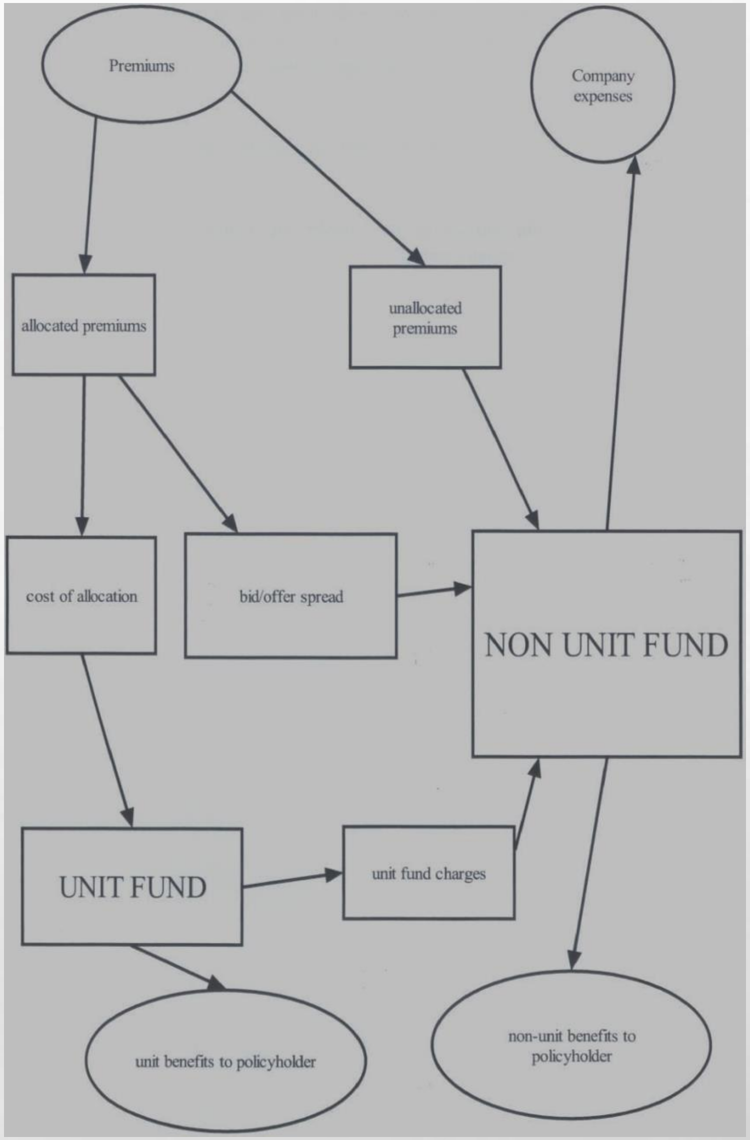
Unallocated premiums are directed to the company’s non-unit fund.

Bid/offer spread is used to help cover expenses and contribute towards profit.

Charges are made from the unit account periodically to cover expenses and benefits (i.e. fund management charge) and may be varied after notice of change given.

Unit-linked contracts may offer guaranteed benefits (e.g. minimum death benefit).

Unit-linked contracts are generally endowment assurance and whole of life contracts.


Premiums
Company expenses
allocated premiums
unallocated premiums
cost of allocation
bid/offer spread
NON UNIT FUND
UNIT FUND
unit fund charges
unit benefits to policyholder
non-unit benefits to policyholder

SURVIVAL MODELS AND LIFE CONTINGENCIES

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LECTURE 29

6.3 Deterministic profit testing for equity-linked insurance

…

6.3.2 Pricing and reserving using profit testing

All procedures are similar to the ones studied with reference to traditional products.

$$cost\ of\ increase\ in\ reserves_t = {}_tV\ (ap)_{x+t-1} - {}_{t-1}V(1+i)$$

or

$$cost\ of\ increase\ in\ reserves_t = increase\ in\ reserves\ - interest\ on\ reserves$$

$$increase\ in\ reserves = {}_tV\ (ap)_{x+t-1} - {}_{t-1}V$$

$$interest\ on\ reserves = i \times {}_{t-1}V$$

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6.3 Deterministic profit testing for equity-linked insurance

…

6.3.2 Pricing and reserving using profit testing

All procedures are similar to the ones studied with reference to traditional products.

Zeroising negative cash flows - Calculating the revised reserves required and the revised profit vector, if negative cash flows other than in Year 1 are to be eliminated

It is a principle of prudent financial management that once sold and funded at the outset a product should be self-supporting. This implies that the profit signature has a single negative value (funds are provided by the insurance company) at policy duration zero. This is often termed “a single financing phase at the outset”.

Many products "naturally" produce profit vectors which usually have a single financing phase. However some products, particularly those with substantial expected outgo at later policy durations, can give profit vectors which have more than one financing phase.

If the contract is to be self funding, then reserves must be established using the earlier positive cash flows. These reserves should be sufficient to pay the later expected negative cash flows.

Example:

The in force expected cash flows for a five-year policy under which no non-unit reserves are held is

(-60.20, -20.50, -17.00, 50.13, 85.75)

Calculate the revised reserves required and the revised profit vector if negative cash flows other than in Year 1 are to be eliminated, assuming that the policy is issued to lives aged 55, mortality is given by

$$q_{55+t} = 0.01 + 0.0017t, \qquad t = 0,1,2,3,4$$

and reserves earn interest at a rate of 5% per annum.

$$(PR_1, \quad PR_2, \quad PR_3, \quad PR_4, \quad PR_5) = (\underbrace{-60.20}_{t=1,x=56}, \underbrace{-20.50}_{t=2,x=57}, \underbrace{-17.00}_{t=3,x=58}, \underbrace{50.13}_{t=4,x=59}, \underbrace{85.75}_{t=5,x=60})$$

Solution:

Negative cash flows are in Years 2 and 3.
If the company can **release money held in reserves** as follows:

20.50 at the end of Year 2

17.00 at the end of Year 3,

then the negative cash flows will be matched by a positive cash flow from reserves and the profit vector will show a zero entry for these two years.

$$(PR_1, \quad PR_2, \quad PR_3, \quad PR_4, \quad PR_5) = (\underbrace{-60.20}_{t=1,x=56}, \underbrace{-20.50}_{t=2,x=57}, \underbrace{-17.00}_{t=3,x=58}, \underbrace{50.13}_{t=4,x=59}, \underbrace{85.75}_{t=5,x=60})$$

No reserves are required after Year 3 since there are no losses. At the start of Year 3, $t = 2$, the required reserve is ${}_2V$

$${}_2V(1.05) = 17.00 \Leftrightarrow {}_2V = 16.19.$$

This means that the total reserve required at the start of Year 2, $t = 1$, is such that

$${}_1V(1.05) = 20.50 + \overbrace{16.19}^{{}_2V}\overbrace{p_{56}}^{*} \Leftrightarrow {}_1V(1.05) = 20.50 + 16.19(1 - 0.0117) \Leftrightarrow {}_1V = 34.76.$$

* Note that that the probability that a policy in force at time 1 is still in force at time 2 ($x = 57$) is $p_{56} = 1 - 0.0117$.

In general, for the entries PR_t to be zeroised

$$ {}_{t-1}V\,(1+i) = -PR_t + {}_tV \times p_{x+t-1} $$

The cash flow at the end of year 1 is then (because the reserve is required for survivors only)

$$PR_1^* = -60.20 + 0 - 34.76p_{55} =$$
$$= -60.20 - 34.76(1 - 0.01) = -94.61.$$

The revised profit vector is (-94.61, 0, 0, 50.13, 85.75)

In general,

$$PR_t^* = PR_t + {}_{t-1}V\,(1 + i) - {}_{t}V \times p_{x+t-1}, t = 1,2, \ldots, n.$$

Exercise:

a) Calculate the non-unit reserves required to zeroise negative cash flows for the in-force expected cash flows

$$(-131.53, -70.11,\ 25, -20.15,\ 55.74, 157.91),$$

for a six-year policy taken out by a 50 year old.

Assume that the probability of death during any year is 0.01 and 6% per annum interest.

b) Calculate the revised profit vector.

c) Repeat a) and b), assuming PR_3=15.

(Revised profit vector: $(-200.34, 0, 0, 0, 55.74, 157.91)$)

$$PR_t^* = PR_t + {}_{t-1}V(1+i) - {}_tV\, p_{x+t-1}, \quad t=1,\dots,n$$

	PR_1	PR_2	PR_3	PR_4	PR_5	PR_6
	-131.53	-70.11	25	-20.15	55.74	157.91
0	1	2	3	4	5	6
(50)	(51)	(52)	(53)	(54)	(55)	(56)

6%

$p_{50} = p_{51} = \dots = p_{55} = 0.99$

Reserve in years 4, 5, 6: ${}_4V = {}_5V\,(= {}_6V) = 0$

To zeroise PR_4 we need a reserve ${}_3V$ so that

$${}_3V(1+i) = -PR_4 + {}_4V\, p_{53} \Leftrightarrow$$

$${}_3V(1.06) = 20.15 + 0 \Leftrightarrow {}_3V = 19.01$$

Since $PR_3 = 25 > 19.01$, this is enough to set up the reserve

To zeroise PR_2 we need a reserve ${}_1V$ so that

$${}_1V(1+i) = 70.11 + {}_2V\, p_{51} \Leftrightarrow$$

$${}_1V(1.06) = 70.11 + 0 \Leftrightarrow {}_1V = 66.14$$

b) Revised profit vector = ?

$PR^*_6 = PR_6 = 157.91$

$PR^*_5 = PR_5 = 55.74$

$PR^*_4 = PR_4 + {}_3V(1+i) - {}_4V\,p_{53} =$

$= -20.15 + 20.15 - 0 = 0$

$PR^*_3 = PR_3 + {}_2V(1+i) - {}_3V\,p_{52} =$

$= 25 + 0 - 19.01 \times 0.99 = 6.18$

$PR^*_2 = PR_2 + {}_1V(1+i) - {}_2V\,p_{51} =$

$= -70.11 + 66.14(1.06) - 0 = 0$

$PR^*_1 = PR_1 + {}_0V(1+i) - {}_1V\,p_{50} =$

$= -131.53 + 0 - 66.14 \times 0.99 = -197.01$

Revised profit vector:

$(-197.01,\ 0,\ 6.18,\ 0,\ 55.74,\ 157.91)$

Summary:

The given examples illustrate that, as already known:

If the premiums for a contract together with all the other data items about the contract are known, then given a basis on which the projected revenue accounts can be calculated, the expected profitability of the contract can be evaluated.

Of course, the actual profitability is an unknown quantity until each respective contract terminates and the actual experience becomes known.

Question:
In developing products the **expected level of profit will usually be specified as an objective.** How can the features of the product be set to achieve this objective?

Answer:
Usually, the benefits and terms and conditions for the payment of these benefits are specified in advance. It follows that only the **level and pattern of premium payments can be varied to meet the profit objective.**

Profit criterion

The objective specified for expected level of profit is termed the "profit criterion". Careful choice of a profit criterion is central.

Examples of the profit criterion are:

NPV of Profit = 40% of Initial Sales Commission
Profit Margin = 3% of NPV of expected premium income

For conventional products, the profit test is completed using a spread sheet or similar software, and the premiums are varied until the required "target" i.e. value of NPV of profit, level of profit margin, is achieved. The premium or price of the product has been determined using a profit test.

Insurers must choose a variety of different assumptions in order to determine how quickly the expected future profit changes on varying any particular assumption. Such alternative bases represent **sensitivity test assumptions.**

The “sensitivity tests” can give the insurer an understanding of how profits might be increased as well as how they might be endangered. The results of these tests may indicate ways in which a product might be re-designed to minimise changes in expected profits. Any redesign would need to be profit tested itself, so this process can be iterative.

In the case of **unit-linked** contracts there is the **additional possibility of varying the charges** and typically this would be the approach taken in pricing such contracts to achieve the profit objective.

For unit-linked products, the management charges are varied to try to achieve an acceptable charging structure (in comparison with other products in the market) which satisfies the profit criterion. The sensitivity of this profit to variation in the key features of the product design e.g. benefits offered, and the assumptions made in determining the expected cash flows e.g. mortality rates, rate of return on investments are usually investigated.

This is done by keeping the premium or charging structure fixed and determining the change in the profit criterion for realistic variation in the characteristics of the product and the assumptions made in the basis. The objective is to design a product which is robust (i.e. profit criterion changes as little as possible) to possible changes in the data and the assumptions used in the profit test.

SURVIVAL MODELS AND LIFE CONTINGENCIES

MASTER IN ACTUARIAL SCIENCE

LECTURE 31

6.4. Stochastic profit testing

Past Exam Questions

1. Explain why a stochastic profit test may reflect reality in a more adequate way than deterministic profit test, for equity-linked insurance.

2. "Financial guarantees are risky and can be expensive." Discuss the use of risk measures to calculate reserves for contracts with financial guarantees.

3. Explain why a stochastic approach would be more (less) appropriate to answer Question 10 (Calculate the level premium so that the sum of the discounted annual emerging surpluses (NPVFP) is 1000 for a 3-year endowment insurance) and Question 11 (Calculate the profit margin on a three-year unit-linked endowment assurance contract).

4. Describe how to use Monte Carlo simulation to perform stochastic profit testing.

5. A Monte Carlo simulation has been performed and a set of 10 000 independent values were sampled from the distribution of the *NPVFP*, for a certain equity linked life product. From this sample, estimates of the mean *m*, standard deviation *s* and other results are shown below.

m	692.50
s	514.05
Minimum	-1223.68
0.01 Quantile	-987.40

Determine a 95% confidence interval for E[*NPV*] and write a brief report with the main conclusions it is possible to derive from the given information

6. Explain why, for a given value of α, the CTE_α reserve is generally more conservative than the Q_α quantile reserve.

7. Discuss briefly this statement: «in developing products the expected level of profit will usually be specified as an objective, termed the ‘profit criterion’. Careful choice of a profit criterion is central».

8. Some products give profit signatures that have more than one financing phase. Explain why the later negative cash flows should be ‘zeroised’ and describe briefly this ‘algorithm’.

9. Comment on the following statement: *“in profit testing the ‘sensitivity tests’ can give the insurer an understanding of how profits might be increased as well as how they might be endangered”*.

6.4. Stochastic profit testing

Using a deterministic profit test does not reflect the reality of the situation adequately in most cases, because the deterministic test, approximately at least, only projects the median result and this may not be enough.

For traditional (conventional) insurance policies it is often assumed that the demographic uncertainty dominates the investment uncertainty – which may be a reasonable assumption if the underlying assets are invested in low risk fixed interest securities of appropriate duration.

The uncertainty involved in equity-linked insurance is very different. The mortality element is assumed diversifiable and is not the major factor. The uncertainty in the investment performance is a far more important element, and it is not diversifiable.

Example:

Selling 1000 equity-linked contracts with GMMBs (guaranteed minimum maturity benefits) to identical lives, for instance, is almost the same as issuing one big contract.

When one policyholder’s fund dips in value, then all dip, increasing the chance that the GMMB will cost the insurer money for every contract.

Profit measures, like the EPV of future profit, do not contain any information about the uncertainty from investment returns.

The profit measure for an equity-linked contract is modelled more appropriately as a random variable rather than a single number. This is achieved by stochastic profit testing, where a sequence of random variables $R_1, R_2, , \dots , R_T$ is introduced, R_t representing the accumulation at time t of a unit amount invested in an equity fund at time $t-1$, so that $R_t - 1$ is the rate of interest earned in the year.

A common assumption for returns on equity portfolios is the independent lognormal assumption, very important in financial modelling, where $R_1, R_2, , \dots , R_T$ are assumed to be mutually independent, and each R_t is assumed to have a lognormal distribution with parameters μ_t and σ^2;

The calculus for stochastic profit testing is the same done in the deterministic profit testing. The difference is that in the stochastic profit testing, the **deterministic investment scenarios are replaced with stochastic scenarios**. The most common practical way to do this is with Monte Carlo simulation.

6.4.1 Monte Carlo simulation

Using Monte Carlo simulation, a large number of outcomes $r_1, r_2, , \ldots, r_T$ for the investment return on the policyholder's fund is generated.

The simulated returns are used in place of the constant investment return assumption in the deterministic case.

The profit test proceeds exactly as described in the deterministic approach, except that **the deterministic test is repeated for each simulated investment return outcome.**

What is the purpose?

The purpose is to generate a random sample of outcomes for the contract, which can be used to determine the probability distribution for each profit measure that might be chosen to assess the product.

That is: to measure the effect of the uncertainty in rates of return, a large number N of sets of rates of return $r_1, r_2, , \ldots, r_T$ is generated and for each set a deterministic profit test is carried.

Illustration with the NPV (modelled as a random variable):

Let NPV_i denote the net present value calculated from the i-th simulation, $i = 1, 2, \ldots, N$. $\{NPV_i\}_{i=1}^{N}$ is a set of N independent values sampled from the distribution of NPV. From this sample it is possible:

To estimate the mean (m), standard deviation (s) and percentiles of the distribution.

To count the number of simulations for which the NPV_i is negative.

To count the number of simulations for which the final fund value is less than the guaranteed benefits, so that there is a liability.

…

It is important whenever reporting summary results from a stochastic simulation to give some measure of the variability of the results, such as a standard deviation or a confidence interval.

Since N is large, the central limit theorem allows to say that a 95% confidence interval for e[npv] is given by

$$\left(m - 1.96\frac{s}{\sqrt{N}}, m + 1.96\frac{s}{\sqrt{N}}\right).$$

6.4.2 Stochastic pricing

Recall that:

1. The equivalence principle premium is defined such that the expected value of the present value of the future loss at the issue of the policy is zero.

2. In fact, the expectation is usually taken over the future lifetime uncertainty (given fixed values for the mortality rates), not the uncertainty in investment returns or non-diversifiable mortality risk.

3. This is an example of an expected value premium principle, where premiums are set considering only the expected value of future loss, not any other characteristics of the loss distribution.

Incorporating a guarantee may add significant risk to a contract and this only becomes clear when modeled stochastically. **The risk cannot be quantified deterministically. Using the mean of the stochastic output is generally not adequate** as it fails to protect the insurer against significant non-diversifiable risk of loss.

For this reason it is not advisable to use the equivalence premium principle when there is significant non-diversifiable risk. Instead we can use **stochastic simulation with different premium principles.**

The quantile premium principle is similar to the portfolio percentile premium principle. It is a principle based on the requirement that the policy should generate a profit with a given probability and can be extended to the pricing of equity-linked policies.

Often it is not possible to determine a premium analytically for equity-linked contracts with certain requirements about profit. However, **one can investigate the effects of changing** the structure of the policy. For instance:

(1) Increasing the premium.

(2) Increasing the annual management charge.

(3) Increasing the expense deductions from the premiums.

(4) Decreasing the GMMB.

(1) Sometimes, increasing the premium makes little difference in terms of the chosen profit criterion. The premium for an equity-linked contract is not like a premium for a traditional contract, since most of it is unavailable to the insurer. The role of the premium in a traditional policy – to compensate the insurer for the risk coverage offered – is taken in equity-linked insurance by the management charge on the policyholder's funds and any loading taken from the premium.

(2,3) Increasing the management charge or the expense loadings does increase the expected net present value but is usually not enough to decrease significantly the probability of a loss.

(4) The one change more effective is reducing the level of the maturity guarantee. This is a demonstration of the important principle that risk management begins with the design of the benefits.

6.4.3 Stochastic reserving (Risk Measures)

Recall that:

1. A profit test can be used to determine the reserves for a conventional life assurance.
2. If the contract is to be self funding, then reserves must be established using the earlier positive cash flows. These reserves should be sufficient to pay the later expected negative cash flows.
3. This requirement is exactly analogous to the need to establish reserves in the non-unit fund for a unit-linked assurance.

Risk Measures to Calculate Reserves

Calculating reserves for policies with significant non-diversifiable risk requires a methodology that takes account of more than just the expected value of the loss distribution. Such methodologies are called **risk measures**.

A risk measure is a functional that is applied to a random loss to give a reserve value that reflects the riskiness of the loss.

There are two common risk measures used to calculate reserves for non-diversifiable risks: the quantile reserve (Value at Risk) and the conditional tail expectation reserve.

6.4.3.1 Quantile Reserve

A quantile reserve is defined in terms of a parameter α, $0 \leq \alpha \leq 1$: The quantile reserve with parameter α represents the amount Q_α which, with probability α, will not be exceeded by the future loss random variable, L.

If L has a continuous distribution function F_α, the α-quantile reserve is Q_α: $Pr(L \leq Q_\alpha) = \alpha \Leftrightarrow Q_\alpha = F_L^{-1}(\alpha)$.

6.4.3.2 Conditional Tail Expectation Reserve

The Conditional Tail Expectation (or CTE) was developed to address some of the problems associated with the quantile risk measure (the quantile reserve assesses the 'worst case' loss but does not take into consideration what the loss will be).

It was proposed more or less simultaneously by several researchers, so it has a number of different names, including Tail Value at Risk (or Tail-VaR), Tail Conditional Expectation (or TCE) and Expected Shortfall.

The CTE_α is the expected loss given that the loss falls in the worst $1-\alpha$ part of the loss distribution, L. The worst $1-\alpha$ part of the loss distribution is the part above the α-quantile, Q_α. then

$$\mathrm{CTE}_\alpha = E\left[L|L > Q_\alpha\right].$$

As the CTE_α is the mean loss given that the loss lies above the VAR at level α, then $\mathrm{CTE}_\alpha \geq Q_\alpha$, and usually strictly greater: for a given value of α, the CTE_α reserve is generally more conservative than the Q_α quantile reserve.

Financial guarantees are risky and can be expensive. Several major life insurance companies have found their solvency at risk through issuing guarantees that were not adequately understood at the policy design stage, and were not adequately reserved for thereafter.

The method of covering that risk by holding a large quantile or CTE reserve reduces the risk, but at great cost in terms of tying up amounts of capital that are huge in terms of the contract overall.

SURVIVAL MODELS AND LIFE CONTINGENCIES

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LECTURE 32

7. Single figure indices to summarise and compare mortality levels *(CT5 CR-15: Mortality, selection and standardisation)*

7.1 Mortality, selection and standardisation

In addition to variation by age and sex, mortality and morbidity rates are observed to vary:

- between geographical areas e.g. countries, regions of a country, urban and rural areas
- by social class e.g. manual and non-manual workers
- over time e.g. mortality rates usually decrease over time

None of these categories e.g. geographical location and time, provide a direct (causal) explanation of the observed differences. Rather they are proxies for the real factors which cause the observed differences.

Such factors are:

- Occupation
- Nutrition
- Housing
- Climate
- Education
- Genetics

It is rare that observed differences in mortality can all be ascribed to a single factor.
It is difficult to separate the effects of different factors. For instance, mortality rates of those living in sub-standard housing are (usually) higher than those of people living in good quality housing.
However, those living in sub-standard housing usually have less well paid occupations and lower educational attainment than those living in good quality housing.
Part or all of the observed difference may be due to these other differences and not to housing differences.

Lower educational attainment $\Longrightarrow$ less well paid occupations $\Longrightarrow$ sub-standard housing $\Longrightarrow$ higher mortality rates

The source of all the following examples and exercises is *CT5 CR-15: Mortality, selection and standardisation.*

Question 15.3

Fat people in the UK tend to come from a "rich" or a "poor" background, rather than an "average wealth" background. Suggest possible reasons for this.

Question 15.4

List four factors that could adversely affect the mortality of a homeless person in a developed country.

How decrements can have a selective effect

One way in which lives in a population can be grouped is by the operation of a decrement (other than death) e.g. retiring on ill-health grounds, getting married, migrating to a new country. Those who do and do not experience this selective decrement will experience different levels of the primary decrement of interest, often mortality or morbidity.

Those getting married usually experience lighter mortality and morbidity than those of the same age who do not get married. Marriage is said to have a selective effect in respect of mortality and morbidity.

Mortality convergence

The variations in mortality are noted most strongly at working ages. These variations can be large and material for insurance companies.

The variation has been seen to continue after retirement but reduces at the very highest ages, although the evidence is disputed. **This convergence of mortality between subgroups at higher ages is referred to as mortality convergence,**

Detailed analysis of mortality convergence is complicated by the low volumes of data at the highest ages.

7.2 Single figure indices

Summary (single figure) mortality indices can be used:

To quantify and compare the mortality experience of different populations;

To monitor the progress over time of a population’s mortality.

The main advantage of the use of single figure indices is their simplicity for summary and comparison, compared to the use of a set of age specific rates.

All summary measures are weighted averages of the age-specific mortality rates or some function of these age-specific rates.

Some indices are particularly designed for comparison with the mortality in a standard population.

Of course, specific features of the underlying mortality rates may be hidden and sometimes extensive data may be required, limiting the situations in which they can be used.

7.2.1 Crude mortality (death) rate

The crude (non-standardised) death rate for a particular population is the total number of deaths observed during the period divided by the total central exposed to risk for the same period. Or the ratio of the total number of deaths in a category to the total exposed to risk in the same category.

$$\text{Crude mortality rate} = \frac{\sum_x \left(E^c_{x,t}\right)\left(m_{x,t}\right)}{\sum_x E^c_{x,t}} = \frac{\text{Actual deaths}}{\text{Total exposed to risk}},$$

$E^c_{x,t}$ = central exposed to risk in population being studied between ages x and $x+t$

$m_{x,t}$ = central rate of mortality in population being studied between ages x and $x+t$

[q_x is the initial rate of mortality. It measures the number of deaths d_x divided by the number of lives alive at age *x*, l_x. The problem is that it assumes that there are l_x persons living between ages *x* and *x*+1; obviously lives will die during the year of age, and will not be exposed to risk for the whole year.

m_x is calculated dividing d_x by the expected number of lives living between ages *x* and *x*+1, equal to $\int_0^1 l_{x+t}\, d_t$.

q_x is the probability a life now aged exactly *x* dies within the next year.

m_x is the probability a life aged anywhere between ages *x* and *x*+1 dies before attaining age *x*+1.]

... differences between mortality rates in using this index to make comparisons of mortality levels between populations.

Consider the following example based on two towns, Youngsville and Oldsville, whose hypothetical populations consist of 100,000 people all aged 20, 40 or 60:

	YOUNGSVILLE			OLDSVILLE		
Age	**Population**	**Deaths**	**Death rate**	**Population**	**Deaths**	**Death rate**
20	70,000	210	0.30%	10,000	15	0.15%
40	20,000	90	0.45%	20,000	60	0.30%
60	10,000	100	1.00%	70,000	525	0.75%
Total	100,000	400	0.40%	100,000	600	0.60%

Looking at the death rates for each age, we see that the mortality rates are higher in Youngsville than in Oldsville at all ages, *ie* any individual in Youngsville is more likely to die at any given age than a corresponding individual in Oldsville. However, the crude death rate is *higher* for Oldsville, which suggests the opposite.

The crude (non-standardised) death rate is easy to calculate, but does not take into account the age or sex structure of the population and gives sometimes misleading results. It is primarily reflecting the average age of the population.

The following three (standardised) indices all endeavour to remove the effect of differing age structures between populations.

7.2.2 Directly Standardised Mortality Rate (DSMR)

The DSMR is defined as the following quotient

$$\frac{\begin{array}{c}\mathit{The\ number\ of\ deaths\ that\ would\ have\ occurred\ in\ a\ standard\ population,}\\ \mathit{had\ the\ mortality\ of\ the\ particular\ population\ applied}\end{array}}{\mathit{The\ total\ central\ exposed\ to\ risk\ of\ the\ standard\ population}}$$

$$\text{DSMR} = \frac{\sum_x \left({}^{s}E_{x,t}^{c}\right)\left(m_{x,t}\right)}{\sum_x {}^{s}E_{x,t}^{c}},$$

${}^{s}E_{x,t}^{c}$ = central exposed to risk for a standard population between ages x and $x+t$

The lower the DSMR, the lighter is the mortality of the particular population, compared to the standard mortality.

Example

Calculate the directly standardised mortality rates for Youngsville and Oldsville.

Standard Population		
Age	Exposed	Deaths
20	40 000	80
40	35 000	140
60	25 000	200
Total	100 000	420
https://portuguesefood.pt/recipe/acorda-alentejana/		

Example

Calculate the directly standardised mortality rates for Youngsville and Oldsville.

The directly standardised mortality rates are therefore:

Youngsville: $\dfrac{527.5}{100{,}000} = 0.005275$ *ie* 0.5275%

Oldsville: $\dfrac{352.5}{100{,}000} = 0.003525$ *ie* 0.3525%

The Directly Standardised Mortality Rate is the mortality rate of a category weighted according to a standard population; that is to say the DSMR takes account of the population structure. Still it is most heavily influenced by the older ages, because the weightings used are based on mortality rates, which will be greatest at the older ages.

It is more complicated to calculate than the crude rate.

Populations may need to be standardised by age, sex or some other risk factor, *e.g*. occupation.

Sex	Occupation	MADEUPTOWN		STANDARD POPULATION	
		Population	Deaths	Population	Deaths
Male	Office worker	20,000	100	10,000	50
	Manual worker	60,000	500	20,000	300
	Other	20,000	250	20,000	500
Female	Office worker	20,000	50	10,000	50
	Manual worker	30,000	200	10,000	100
	Other	50,000	900	30,000	500
Total		200,000	2,000	100,000	1,500

Calculate the directly standardised mortality rate for Madeuptown, standardising by,

(a) occupation,

(b) sex, and

(c) occupation *and* sex.

The crude mortality rate is: $2{,}000/200{,}000 = 0.01$ *ie* 1%.

(a) *Standardising by occupation*

Standardising by occupation (by calculating how many people and deaths there would have been if the population had conformed to the same proportions as the standard population):

So the death rate standardised by occupation is: $2{,}259.5/200{,}000 = 0.0113$

ie 11.3 per 1,000.

(b) *Standardising by sex*

Since the proportions of each sex are the same as in the standard population, standardisation by sex makes no difference to the crude mortality rate.

(c) *Standardising by occupation and sex* (they are both 0.01)

So the death rate standardised by sex and occupation is: 2,196.7/200,000 = 0.0110 *ie* 11.0 per 1,000.

The crude death rate is lower than both the death rate standardised by occupation alone and the death rate standardised by occupation and sex. Hence Madeuptown has a lower prevalence of the high mortality occupations. Standardising by sex as well as occupation reduces the death rate, which suggests that Madeuptown has a higher prevalence of the weaker sex in the high mortality occupations.

7.2.3 **Indirectly standardised mortality rate (ISMR)**

The ISMR is a good approximation to the DSMR. It is defined as the following quotient

$$\frac{Crude\ mortality\ rate\ for\ standard\ population}{\dfrac{Expected\ deaths\ in\ population}{Actual\ deaths\ in\ population}}$$

$$\text{ISMR} = \frac{\dfrac{\sum_x \left({}^{s}E^{c}_{x,t}\right)\left({}^{s}m_{x,t}\right)}{\sum_x {}^{s}E^{c}_{x,t}}}{\dfrac{\sum_x \left(E^{c}_{x,t}\right)\left({}^{s}m_{x,t}\right)}{\sum_x \left(E^{c}_{x,t}\right)\left(m_{x,t}\right)}},$$

${}^{s}m_{x,t}$ = central rate of mortality in standard population between ages x and $x+t$

$$\text{ISMR} = \frac{\frac{\sum_x \left({}^{S}E^{C}_{x,t}\right)\left({}^{S}m_{x,t}\right)}{\sum_x {}^{S}E^{C}_{x,t}}}{\frac{\sum_x \left(E^{C}_{x,t}\right)\left({}^{S}m_{x,t}\right)}{\sum_x \left(E^{C}_{x,t}\right)\left(m_{x,t}\right)}} \text{ can be decomposed as}$$

$$\text{ISMR} = F \times Crude\ \ mortality\, rate\ for\ population,$$

where

$$\text{F} = \frac{\frac{\sum_x \left({}^{S}E^{C}_{x,t}\right)\left({}^{S}m_{x,t}\right)}{\sum_x {}^{S}E^{C}_{x,t}}}{\frac{\sum_x \left(E^{C}_{x,t}\right)\left({}^{S}m_{x,t}\right)}{\sum_x E^{C}_{x,t}}} = \frac{\text{Crude mortality rate for standard population}}{\text{Crude mortality rate for population, using standard mortality}}$$

is the Area Comparability **F**actor.

Example

Calculate the area comparability factor for Youngsville using the data for the standard population in Section 9.1.

Solution

We have already seen that the mortality rate based on the standard population and standard population mortality is 0.42%.

The mortality rate based on the regional population and standard population mortality is:

$$70\% \times 0.20 + 20\% \times 0.40 + 10\% \times 0.80 = 0.30\%$$

So the approximate area comparability factor is: $0.42\%/0.30\% = 1.400$

Question 15.19

Calculate the area comparability factor and indirectly standardised mortality rate for Oldsville.

So the approximate area comparability factor is: $0.42\%/0.66\% = 0.636$

The indirectly standardised mortality rate is: $0.636 \times 0.6\% = 0.38\%$

Question 15.20

Calculate a standardised mortality rate for Madeuptown using indirect standardisation by both occupation and sex.

So the approximate area comparability factor is: $1.50\%/1.37\% = 1.098$

The indirectly standardised mortality rate is calculated by applying this to the crud

mortality rate: $1.098 \times \dfrac{2,000}{200,000} = 1.10\%$

Question 15.20

Calculate a standardised mortality rate for Madeuptown using indirect standardisation by both occupation and sex.

So the approximate area comparability factor is: $1.50\%/1.37\% = 1.098$

The indirectly standardised mortality rate is calculated by applying this to the crud

mortality rate: $1.098 \times \dfrac{2,000}{200,000} = 1.10\%$

F provides information about the structure of the population being studied, relative to the standard population.

A value of F less than 1 indicates that the population structure is more heavily weighted towards individuals who experience heavier mortality (older ages or males).

In short, Indirectly Standardised Mortality Rate is an approximation to the Directly Standardised Mortality Rate being the crude rate for the standard population multiplied by the ratio of actual to expected deaths for the region. This is the same as the crude rate for the local population multiplied by the Area Comparability Factor.

The ISMR does not require local records of births to be analysed by age grouping, which is an advantage over the DSMR. Very often it is possible to compute both and they are similar, so the approximation is usually acceptable.

Further conclusions:

- if DSMR and ISMR and crude rate for the standard population > crude rate for the studied population

then the reason for the low crude rate compared to the standard population is due to population distribution by age.

- if crude rate for the standard population > DSMR and ISMR

then crude rate for the studied population is lower, even allowing for the age distribution.

Summary:

Crude mortality rate: the ratio of the total number of deaths in a category to the total exposed to risk in the same category.

Directly standardised mortality rate: the mortality rate of a category weighted according to a standard population.

Indirectly standardised mortality rate: an approximation to the directly standardised mortality rate being the crude rate for the standard population multiplied by the ratio of actual to expected deaths.

7.2.4 5 Standardised mortality ratio (SMR)

The SMR is defined as

$$\frac{\text{The number of deaths observed in the particular population}}{\begin{array}{c}\text{The number of deaths that would have occurred in the particular population}\\ \text{had the mortality of the standard population applied}\end{array}}$$

$$\text{or SMR} = \frac{\text{Actual deaths in population}}{\text{Expected deaths in population}} = \frac{\sum_x \left(E_{x,t}^c\right)\left(m_{x,t}\right)}{\sum_x \left(E_{x,t}^c\right)\left({}^s m_{x,t}\right)}$$

The SMR compares the indirectly standardized mortality rate with the crude mortality rate in the standard population.

Values less than 1 indicate populations with mortality lighter than that in the standard population.

Question 15.21

Calculate the standardised mortality ratio for Oldsville

Solution 15.21

The actual number of deaths in Oldsville is 600.

The expected number of deaths is:

$$0.20\% \times 10{,}000 + 0.40\% \times 20{,}000 + 0.80\% \times 70{,}000 = 660$$

So the indirectly standardised mortality ratio (SMR) is $600/660 = 0.909$

Mortality levels for a certain country have been studied at national and regional level. Explain the circumstances under which a particular region may have an Area Comparability Factor of 0.5.

(i) Discuss the suitability of the crude death rate, the standardised mortality rate and the standardised mortality ratio for comparing

(a) the mortality, at different times, of the population of a given country

(b) the mortality, at a certain time, of two different occupational groups in the same population

(ii) The following table gives a summary of mortality for one of the occupational groups and for the country as a whole.

	Occupation A		**Whole Country**	
Age group	*Exposed to risk*	*Deaths*	*Exposed to risk*	*Deaths*
20–34	15,000	52	960,000	3,100
35–49	12,000	74	1,400,000	7,500
50–64	10,000	109	740,000	7,100
	37,000	235	3,100,000	17,700

Calculate the crude death rate, the standardised mortality rate and tl standardised mortality ratio for Occupation A.